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Luigi Cremona

F. GUTEKUNST, PHILA.

The Parametric Representation of the Tetrahedroid Surface.

By Derrick N. Lehmer.

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2. We may represent the tetrahedroid surface parametrically as follows:§

$$
\begin{aligned}
x_0 &= \sigma\, u\, \bar{\sigma}\, v, \\
x_1 &= \sigma_1 u\, \bar{\sigma}_1 v, \\
x_2 &= \sigma_2 u\, \bar{\sigma}_2 v, \\
x_3 &= \sigma_3 u\, \bar{\sigma}_3 v.
\end{aligned}
$$

x_0, x_1, x_2, x_3 are the four homogeneous coordinates of a point in space. u and v are independent parameters $\sigma_\lambda\, \bar{\sigma}_\lambda$ are the σ_λ-functions of Weierstrass built on

* Weber, Crelle's Journal, Vol. 84, p. 349.

† Cayley, Collected Works, Vol. I, p. 302, Vol. V, p. 431 and Vol. X, p. 487.

‡ Study, "Sphärische Trigonometrie, Orthogonal Substitutionen und Elliptische Functionen," p. 225.

‖ Lacour, Nouvelles Annales, XVII (3), p. 266.

§ See Hutchinson's Thesis On the Reduction of Hyperelliptic Functions, p. 36. Also Bricard, Nouvelles Annales, 3d series, Vol. 18, p. 197, 1899.

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independent invariants. To σu belong e_1, e_2, e_3, and the periods $2\omega, 2\omega'$. To $\bar{\sigma} v$ belong $\bar{e}_1, \bar{e}_2, \bar{e}_3$, and the periods $2\bar{\omega}, 2\bar{\omega}'$.

We shall further use the notations

$$\begin{aligned} \omega &= \omega_1, & \bar{\omega} &= \bar{\omega}_1, \\ \omega + \omega' &= \omega_2, & \bar{\omega} + \bar{\omega}' &= \bar{\omega}_2, \\ \omega' &= \omega_3, & \bar{\omega}' &= \bar{\omega}_3, \end{aligned}$$

also

$$\begin{aligned} a^2 &= e_2 - e_3, & \bar{a}^2 &= \bar{e}_2 - \bar{e}_3, \\ b^2 &= e_3 - e_1, & \bar{b}^2 &= \bar{e}_3 - \bar{e}_1, \\ c^2 &= e_1 - e_2, & \bar{c}^2 &= \bar{e}_1 - \bar{e}_2. \end{aligned}$$

3. If we put $\dfrac{x_1}{x_0} = x, \quad \dfrac{x_2}{x_0} = y, \quad \dfrac{x_3}{x_0} = z,$
we get*

$$\begin{aligned} x^2 &= (\wp u - e_1)(\bar{\wp} v - \bar{e}_1), \\ y^2 &= (\wp u - e_2)(\bar{\wp} v - \bar{e}_2), \\ z^2 &= (\wp u - e_3)(\bar{\wp} v - \bar{e}_3), \end{aligned}$$

whence solving for $\wp u$, $\bar{\wp} v$ and $\wp u \bar{\wp} v$ we get

$$\begin{aligned} \wp u &= \phi(x, y, z), \\ \bar{\wp} v &= \psi(x, y, z), \\ \wp u \bar{\wp} v &= \chi(x, y, z), \end{aligned}$$

where ϕ, ψ, and χ are functions of x, y, z of the form

$$Ax^2 + By^2 + Cz^2 + D.$$

The equation of the surface is then

$$\chi = \phi\psi.$$

The surface is thus seen to be of the fourth degree. The Cartesian equation may be obtained in a simpler way by using the results of §6. If u is fixed while v varies the point (x, y, z) lies on the intersection of two quadrics, and similarly when v is fixed while u varies. The "parametric lines" are therefore elliptic twisted quartics.

4. It is important to determine the region in which u and v are to vary in order to obtain the whole surface. In other words we must determine what

* See Schwarz, Formeln und Lehrsätze, Art. 18.

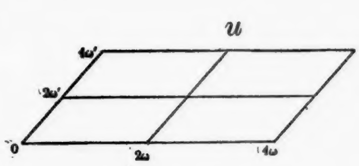
values of u, v correspond to a given value of x, y, z. From the preceding paragraph

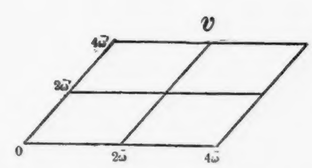
$$\wp u = \phi(x, y, z),$$
$$\bar{\wp} v = \psi(x, y, z),$$

so for given values x_0, y_0, z_0 of x, y, z we get two sets of values;

$$u = \pm u_0 + 2\mu\omega + 2\mu'\omega',$$
$$v = \pm v_0 + 2\nu\bar{\omega} + 2\nu'\bar{\omega}'.$$

where u_0, v_0 are any pair of values of u, v which give x_0, y_0, z_0 while μ, μ', ν, ν' are integers. It is not necessary to consider values of u and v outside the four parallelograms as shown

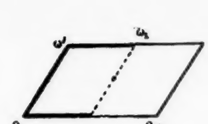


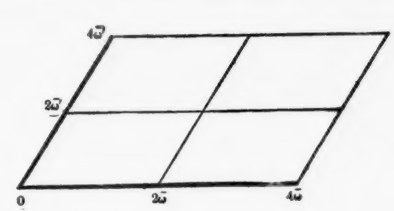


because 4ω, $4\omega'$ and $4\bar{\omega}$, $4\bar{\omega}'$ are periods for all the coordinates. Now ϕ and ψ were functions involving only the squares of x, y, z, so all the values

$$\pm u_0 + 2\mu\omega + 2\mu'\omega',$$
$$\pm v_0 + 2\nu\bar{\omega} + 2\nu'\bar{\omega}',$$

of u and v will give correct values of x_0, y_0, z_0 to sign *près*. It is found that by applying formulæ 7 of article 18, Schwarz "Formeln," that if we take any one of the eight possible values of u and combine it with all the eight possible values of v lying in the four parallelograms above we get all the eight possibilities of sign. We may thus arbitrarily choose our u in the lower half of the first parallelogram and allow v all the four parallelograms. As to the points on the boundaries in the v plane the usual conventions apply and we may omit the upper and right hand boundaries altogether. In the u plane the only boundary necessary is from the origin up to and including the points ω and ω' and from ω' up to and including $\omega + \omega' = \omega_2$.





4. It is important to note the effect of certain transformations of the form

$$u' = u + \varepsilon\omega_\lambda,$$
$$v' = v + \varepsilon\bar{\omega}_\lambda + 2\varkappa\bar{\omega}_i,$$

$\varepsilon = 0$ or 1, $\lambda = 1, 2, 3$, $i = 1, 2, 3$, $\varkappa =$ an integer. Restricting ourselves to the range of values indicated in the preceding paragraph we have the following sixteen transformations:

1°.	$u' = u,$	$v' = v,$
2°.	$u' = u,$	$v' = v + 2\bar{\omega}_1,$
3°.	$u' = u,$	$v' = v + 2\bar{\omega}_2,$
4°.	$u' = u,$	$v' = v + 2\bar{\omega}_3,$
5°.	$u' = u + \omega_1,$	$v' = v + \bar{\omega}_1,$
6°.	$u' = u + \omega_1,$	$v' = v + \bar{\omega}_1 + 2\bar{\omega}_1,$
7°.	$u' = u + \omega_1,$	$v' = v + \bar{\omega}_1 + 2\bar{\omega}_2,$
8°.	$u' = u + \omega_1,$	$v' = v + \bar{\omega}_1 + 2\bar{\omega}_3,$
9°.	$u' = u + \omega_2,$	$v' = v + \bar{\omega}_2,$
10°.	$u' = u + \omega_2,$	$v' = v + \bar{\omega}_2 + 2\bar{\omega}_1,$
11°.	$u' = u + \omega_2,$	$v' = v + \bar{\omega}_2 + 2\bar{\omega}_2,$
12°.	$u' = u + \omega_2,$	$v' = v + \bar{\omega}_2 + 2\bar{\omega}_3,$
13°.	$u' = u + \omega_3,$	$v' = v + \bar{\omega}_3,$
14°.	$u' = u + \omega_3,$	$v' = v + \bar{\omega}_3 + 2\bar{\omega}_1,$
15°.	$u' = u + \omega_3,$	$v' = v + \bar{\omega}_3 + 2\bar{\omega}_2,$
16°.	$u' = u + \omega_3,$	$v' = v + \bar{\omega}_3 + 2\bar{\omega}_3,$

The effect of these transformations on the coordinates is exhibited in the following table which is easily constructed by applying formulæ (2) of article 22, and formulæ 6 of article 18, Schwarz, "Formeln."

	x_0	x_1	x_2	x_3
1°	x_0	x_1	x_2	x_3
2°	$-x_0$	$-\quad x_1$	x_2	x_3
3°	$-x_0$	x_1	$-\quad x_2$	x_3
4°	$-x_0$	x_1	x_2	$-\quad x_3$
5°	x_1	$-b\bar{b}c\bar{c}\,x_0$	$c\bar{c}\,x_3$	$b\bar{b}\,x_2$
6°	$-x_1$	$b\bar{b}c\bar{c}\,x_0$	$c\bar{c}\,x_3$	$b\bar{b}\,x_2$
7°	$-x_1$	$-b\bar{b}c\bar{c}\,x_0$	$-\quad c\bar{c}\,x_3$	$b\bar{b}\,x_2$
8°	$-x_1$	$-b\bar{b}c\bar{c}\,x_0$	$c\bar{c}\,x_3$	$-\quad b\bar{b}\,x_2$
9°	x_2	$c\bar{c}\,x_3$	$-c\bar{c}a\bar{a}\,x_0$	$a\bar{a}\,x_1$
10°	$-x_2$	$-\quad c\bar{c}\,x_3$	$-c\bar{c}a\bar{a}\,x_0$	$a\bar{a}\,x_1$
11°	$-x_2$	$c\bar{c}\,x_3$	$c\bar{c}a\bar{a}\,x_0$	$a\bar{a}\,x_1$
12°	$-x_2$	$c\bar{c}\,x_3$	$-c\bar{c}a\bar{a}\,x_0$	$-\quad a\bar{a}\,x_1$
13°	x_3	$b\bar{b}\,x_2$	$a\bar{a}\,x_1$	$-a\bar{a}bb\,x_0$
14°	$-x_3$	$-\quad b\bar{b}\,x_2$	$a\bar{a}\,x_1$	$-a\bar{a}b\bar{b}\,x_0$
15°	$-x_3$	$b\bar{b}\,x_2$	$-\quad a\bar{a}\,x_1$	$-a\bar{a}b\bar{b}\,x_0$
16°	$-x_3$	$b\bar{b}\,x_2$	$a\bar{a}\,x_1$	$a\bar{a}b\bar{b}\,x_0$

5. It is not difficult to see that the above sixteen transformations form a group, and it follows that when we have found a relation between any or all of the coordinates we can at once derive sixteen other relations by operating on these coordinates with these transformations. The relations thus found need

not necessarily be distinct. Thus the equation of the surface will, of course, be invariant under all these transformations.

6. THEOREM. *The intersection of the surface with each of the planes of reference is a pair of conics. The three vertices of the tetrahedron of reference in any reference plane furnishes a triangle which is self-polar to both the conics in that plane.*

To prove this, put $x_0 = 0$. This gives $u = 0$ or $v = 0$. Taking $u = 0$, we have

$$\begin{aligned} x_0 &= 0 \quad , \\ x_1 &= \bar{\sigma}_1 v, \\ x_2 &= \bar{\sigma}_2 v, \\ x_3 &= \bar{\sigma}_3 v, \end{aligned}$$

whence

$$a^2x_1^2 + b^2x_2^2 + c^2x_3^2 = a^2\bar{\sigma}_1^2 v + b^2\bar{\sigma}_2^2 v + c^2\sigma_3^2 v = 0$$

(Schwarz, "Formeln," article 24.) Our conic is thus

$$a^2x_1^2 + b^2x_2^2 + c^2x_3^2 = 0.$$

The other conic in this plane is obtained by putting $v = 0$ and turns out to be

$$\bar{a}^2x_1^2 + \bar{b}^2x_2^2 + \bar{c}^2x_3^2 = 0.$$

The sections by the three other coordinate planes are obtained by operating on the above with the transformations of our group. The sections by the four planes are

$$\begin{aligned}
x_0 &= 0, & (a^2x_1^2 + b^2x_2^2 + c^2x_3^2)(\bar{a}^2x_1^2 + \bar{b}^2x_2^2 + \bar{c}^2x_3^2) &= 0, \\
x_1 &= 0, & (a^2\bar{b}^2\bar{c}^2x_0^2 + \bar{b}^2x_2^2 + \bar{c}^2x_3^2)(\bar{a}^2b^2c^2x_0^2 + b^2x_2^2 + c^2x_3^2) &= 0, \\
x_2 &= 0, & (\bar{a}^2b^2\bar{c}^2x_0^2 + \bar{a}^2x_1^2 + \bar{c}^2x_3^2)(a^2\bar{b}^2c^2x_0^2 + a^2x_1^2 + c^2x_3^2) &= 0, \\
x_3 &= 0, & (\bar{a}^2\bar{b}^2c^2x_0^2 + \bar{a}^2x_1^2 + \bar{b}^2x_2^2)(a^2b^2\bar{c}^2x_0^2 + a^2x_1^2 + b^2x_2^2) &= 0,
\end{aligned}$$

These conics are seen to be referred to their self-polar triangles.

7. By means of the intersections of the surface with the planes of reference given above we may easily derive the equation of the surface in tetrahedral coordinates.

From paragraph 3 we note that the equation is of the form

$$\sum_{i,j=0,1,2,3} A_{ij}x_i^2x_j^2 = 0.$$

Put successively $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ in this equation and identify with the above intersections. The equation of the surface thus obtained is

$$
\begin{aligned}
a^2\bar{a}^2 b^2\bar{b}^2 c^2\bar{c}^2 x_0^4 &+ a^2\bar{a}^2 x_1^4 + b^2\bar{b}^2 x_2^4 + c^2\bar{c}^2 x_3^4 \\
&+ a^2\bar{a}^2 (c^2\bar{b}^2 + \bar{c}^2 b^2)\, x_0^2 x_1^2 \\
&+ b^2\bar{b}^2 (a^2\bar{c}^2 + \bar{a}^2 c^2)\, x_0^2 x_2^2 \\
&+ c^2\bar{c}^2 (b^2\bar{a}^2 + \bar{b}^2 a^2)\, x_0^2 x_3^2 \\
&+ \qquad (a^2\bar{b}^2 + \bar{a}^2 b^2)\, x_1^2 x_2^2 \\
&+ \qquad (b^2\bar{c}^2 + \bar{b}^2 c^2)\, x_2^2 x_3^2 \\
&+ \qquad (c^2\bar{a}^2 + \bar{c}^2 a^2)\, x_3^2 x_1^2 = 0.
\end{aligned}
$$

Study has expressed this in a somewhat more elegant form by writing

$$
\begin{aligned}
x_0' &= -\sqrt{a\bar{a}\, b\bar{b}\, c\bar{c}}\; x_0, \\
x_1' &= \sqrt{a\bar{a}}\; x_1, \\
x_2 &= \sqrt{b\bar{b}}\; x_2, \\
x_3 &= \sqrt{c\bar{c}}\; x_3, \\
a/a &= a, \\
b/\bar{b} &= b, \\
c/\bar{c} &= c.
\end{aligned}
$$

We thus get, dropping accents,

$$
\begin{aligned}
x_0^4 + x_1^4 + x_2^4 + x_3^4 &+ \left(\frac{c}{b} + \frac{b}{c}\right)(x_0^2x_1^2 + x_2^2x_3^2) \\
&+ \left(\frac{a}{c} + \frac{c}{a}\right)(x_0^2x_2^2 + x_3^2x_1^2) \\
&+ \left(\frac{a}{b} + \frac{b}{a}\right)(x_0^2x_3^2 + x_1^2x_2^2) = 0.
\end{aligned}
$$

From this, the equation of Fresnel's "Wave Surface" may be easily derived by the transformation

$$
\begin{aligned}
x &= \sqrt{-ab}\;\frac{x_1}{x_0}, \\
y &= \sqrt{-bc}\;\frac{x_2}{x_0}, \\
z &= \sqrt{-ca}\;\frac{x_3}{x_0}.
\end{aligned}
$$

8. *Singular Planes.* Starting from the identities (Schwarz, "Formeln," article 24),

$$a^2\sigma_1^2 u + b^2\sigma_2^2 u = -c^2\sigma_3^2 u,$$
$$\bar{a}^2\bar{\sigma}_1^2 v + \bar{b}^2\bar{\sigma}_2^2 v = -\bar{c}^2\bar{\sigma}_3^2 v,$$

and using the well-known formula

$$(P^2+Q^2)(R^2+S^2) = (PR+QS)^2 + (PS-QR)^2,$$

we get

$$(a\bar{a}\sigma_1 u\bar{\sigma}_1 v + b\bar{b}\sigma_2 u\bar{\sigma}_2 v)^2 + (a\bar{b}\sigma_1 u\bar{\sigma}_2 v - \bar{a}b\bar{\sigma}_1 v\sigma_2 u)^2 = c^2\bar{c}^2\sigma_3^2 u\bar{\sigma}_3^2 v$$

or

$$c^2\bar{c}^2\sigma_3^2 u\bar{\sigma}_3^2 v - (a\bar{a}\sigma_1 u\bar{\sigma}_1 v + b\bar{b}\sigma_2 u\bar{\sigma}_2 v)^2 = (a\bar{b}\sigma_1 u\bar{\sigma}_2 v - \bar{a}b\bar{\sigma}_1 v\sigma_2 u)^2,$$

whence,

$$(c\bar{c}\sigma_3 u\bar{\sigma}_3 v - a\bar{a}\sigma_1 u\bar{\sigma}_1 v - b\bar{b}\sigma_2 u\bar{\sigma}_2 v)(c\bar{c}\sigma_3 u\bar{\sigma}_3 v + a\bar{a}\sigma_1 u\bar{\sigma}_1 v + b\bar{b}\sigma_2 u\bar{\sigma}_2 v)$$
$$= (a\bar{b}\sigma_1 u\bar{\sigma}_2 v - \bar{a}b\bar{\sigma}_1 v\sigma_2 u)^2.$$

Now, the identity which we started with holds also when $\sigma_2 u$ is changed to $-\sigma_2 u$. We thus get the identity

$$(c\bar{c}\sigma_3 u\bar{\sigma}_3 v - a\bar{a}\sigma_1 u\bar{\sigma}_1 v + b\bar{b}\sigma_2 u\bar{\sigma}_2 v)(c\bar{c}\sigma_3 u\bar{\sigma}_3 v + a\bar{a}\sigma_1 u\bar{\sigma}_1 v - b\bar{b}\sigma_2 u\bar{\sigma}_2 v)$$
$$= (a\bar{b}\sigma_1 u\bar{\sigma}_2 v + \bar{a}b\bar{\sigma}_1 v\sigma_2 u).$$

Multiplying these last two equations together, member by member, we have

$$(c\bar{c}x_3 - a\bar{a}x_1 - b\bar{b}x_2)(c\bar{c}x_3 + a\bar{a}x_1 + b\bar{b}x_2)(c\bar{c}x_3 - a\bar{a}x_1 + b\bar{b}x_2)(c\bar{c}x_3 + a\bar{a}x_1 - b\bar{b}x_2)$$
$$= (a^2\bar{b}^2\sigma_1^2 u\bar{\sigma}_2^2 v - \bar{a}^2 b^2\bar{\sigma}_1^2 v\sigma_2^2 u)^2.$$

We may identify this equation of the tetrahedroid with that given in paragraph 7. Putting the right side of this last equation in the form

$$(A_0 x_0^2 + A_1 x_1^2 + A_2 x_2^2 + A_3 x_3^2)^2$$

we get, comparing coefficients,

$$\frac{A_0^2}{a^2\bar{a}^2 b^2\bar{b}^2 c^2\bar{c}^2} = \frac{A_1^2 - a^4\bar{a}^4}{a^2\bar{a}^2} = \frac{A_2^2 - b^4\bar{b}^4}{b^2\bar{b}^2} = \text{etc.} = \rho.$$

By using the relations $a^2 + b^2 + c^2 = 0$, $\bar{a}^2 + \bar{b}^2 + \bar{c}^2 = 0$, it is not difficult to obtain the value of ρ in the simple form

$$\rho = \frac{4a^2\bar{a}^2 b^2\bar{b}^2 c^2\bar{c}^2}{(\bar{b}^2 c^2 - b^2\bar{c}^2)^2},$$

whence,

$$A_0 = \frac{2a^2\bar{a}^2b^2\bar{b}^2c^2\bar{c}^2}{\bar{b}^2c^2 - b^2\bar{c}^2},$$

$$A_1 = a^2\bar{a}^2\,\frac{\bar{b}^2c^2 + b^2\bar{c}^2}{\bar{b}^2c^2 - b^2\bar{c}^2},$$

$$A^2 = b^2\bar{b}^2\,\frac{\bar{c}^2a^2 + c^2\bar{a}^2}{\bar{b}^2c^2 - b^2\bar{c}^2},$$

$$A_3 = c^2\bar{c}^2\,\frac{\bar{a}^2b^2 + a^2\bar{b}^2}{\bar{b}^2c^2 - b^2\bar{c}^2}.$$

We have thus thrown the equation of the surface in the form

$$q_1q_2q_3q_4 = Q_0^2,$$

where

$$\begin{aligned} q_1 &= \quad a\bar{a}x_1 + b\bar{b}x_2 + c\bar{c}x_3, \\ q_2 &= -a\bar{a}x_1 + b\bar{b}x_2 + c\bar{c}x_3, \\ q_3 &= -a\bar{a}x_1 - b\bar{b}x_2 + c\bar{c}x_3, \\ q_4 &= \quad a\bar{a}x_1 - b\bar{b}x_2 + c\bar{c}x_3, \end{aligned}$$

and

$$Q_0 = \frac{1}{\bar{b}^2c^2 - b^2\bar{c}^2}\Big[2a^2\bar{a}^2b^2\bar{b}^2c^2\bar{c}^2x_0^2 + a^2\bar{a}^2(\bar{b}^2c^2 + b^2\bar{c}^2)\,x_1^2 + b^2\bar{b}^2(\bar{c}^2a^2 + c^2\bar{a}^2)\,x_2^2 + c^2\bar{c}^2(\bar{a}^2b^2 + a^2\bar{b}^2)\,x_3^2\Big]$$

From the general theory of surfaces, each of the planes $q_i = 0$ is tangent to the tetrahedroid along its intersection with the quadric surface $Q_0 = 0$. We have, in fact, found four singular tangent planes which touch the tetrahedroid along a conic counted twice.

Apply, now, the operations of the group. We thus derive sixteen singular tangent planes, all of which touch the surface along a conic counted twice. Writing, in general, $q\,(a, b, c, d)$ for the plane $ax_0 + bx_1 + cx_2 + dx_3 = 0$, our singular tangent planes are

$$\begin{aligned} &q\,(\;0, \quad a\bar{a}, \quad \pm b\bar{b}, \quad \pm c\bar{c}), \\ &q\,(a\bar{a}, \quad 0, \quad \pm 1, \quad \pm 1), \\ &q\,(b\bar{b}, \quad \pm 1, \quad 0, \quad \pm 1), \\ &q\,(c\bar{c}, \quad \pm 1, \quad \pm 1, \quad 0). \end{aligned}$$

The four quadric surfaces Q_0, Q_1, Q_2, Q_3 are seen to be referred to their self-polar tetrahedrons. They are:

$$\begin{aligned}
Q_0 \equiv 2a^2\bar{a}^2b^2\bar{b}^2c^2\bar{c}^2x_0^2 &+ a^2\bar{a}^2(\bar{b}^2c^2+b^2\bar{c}^2)\,x_1^2 \\
&+ b^2\bar{b}^2(\bar{c}^2a^2+c^2\bar{a}^2)\,x_2^2 \\
&+ c^2\bar{c}^2(\bar{a}^2b^2+a^2\bar{b}^2)\,x_3^2 = 0, \\
Q_1 \equiv \qquad 2a^2\bar{a}^2x_1^2 &+ a^2\bar{a}^2(\bar{b}^2c^2+b^2\bar{c}^2)\,x_0^2 \\
&+ \qquad (\bar{c}^2a^2+c^2\bar{a}^2)\,x_3^2 \\
&+ \qquad (\bar{a}^2b^2+a^2\bar{b}^2)\,x_2^2 = 0, \\
Q_2 \equiv \qquad 2b^2\bar{b}^2x_2^2 &+ \qquad (\bar{b}^2c^2+b^2\bar{c}^2)\,x_3^2 \\
&+ b^2\bar{b}^2(\bar{c}^2a^2+c^2\bar{a}^2)\,x_0^2 \\
&+ \qquad (\bar{a}^2b^2+a^2\bar{b}^2)\,x_1^2 = 0, \\
Q_3 \equiv \qquad 2c^2\bar{c}^2x_3^2 &+ \qquad (\bar{b}^2c^2+b^2\bar{c}^2)\,x_2^2 \\
&+ \qquad (\bar{c}^2a^2+c^2\bar{a}^2)\,x_1^2 \\
&+ c^2\bar{c}^2(\bar{a}^2b^2+a^2\bar{b}^2)\,x_0^2 = 0.
\end{aligned}$$

The above method of obtaining the singular planes of the surface is used by Lacour in his treatment of Fresnel's "Wave Surface" (Nouvelles Annales, third series, Vol. XVII, p. 266).

A glance at the equation of the tetrahedroid shows that

$$\begin{aligned}
x_0\,Q_0 &= \frac{\partial f}{\partial x_0}, \\
x_1\,Q_1 &= \frac{\partial f}{\partial x_1}, \\
x_2\,Q_2 &= \frac{\partial f}{\partial x_2}, \\
x_3\,Q_3 &= \frac{\partial f}{\partial x_3}.
\end{aligned}$$

The tangent plane to the tetrahedroid is therefore given parametrically as follows:

$$\begin{aligned}
\rho u_0 &= x_0'\,Q_0', \\
\rho u_1 &= x_1'\,Q_1', \\
\rho u_2 &= x_2'\,Q_2', \\
\rho u_3 &= x_3'\,Q_3',
\end{aligned}$$

or in terms of u and v,

$$\begin{aligned}
\rho u_0 &= \sigma u \bar{\sigma} v \, (a^2 \bar{b}^2 \sigma_1^2 u \bar{\sigma}_2^2 v - \bar{a}^2 b^2 \bar{\sigma}_1^2 v \sigma_2^2 u), \\
\rho u_1 &= \sigma_1 u \bar{\sigma}_1 v \, (\bar{a}^2 \sigma_2^2 u \bar{\sigma}^2 v - a^2 \sigma^2 u \bar{\sigma}_2^2 v) \quad , \\
\rho u_2 &= \sigma_2 u \bar{\sigma}_2 v \, (\bar{b}^2 \sigma_3^2 u \bar{\sigma}^2 v - b^2 \sigma^2 u \bar{\sigma}_3^2 v) \quad , \\
\rho u_3 &= \sigma_3 u \bar{\sigma}_3 v \, (\bar{c}^2 \sigma_1^2 u \bar{\sigma}^2 v - c^2 \sigma^2 u \bar{\sigma}_1^2 v) \quad .
\end{aligned}$$

By using the formulæ of art. 24, Schwarz, " Formeln," these equations may be written

$$\begin{aligned}
\rho u_0 &= \sigma u \bar{\sigma} v \, (a^2 \bar{b}^2 \sigma_1^2 u \bar{\sigma}_2^2 v - \bar{a}^2 b^2 \sigma_2^2 u \bar{\sigma}_1^2 v), \\
\rho u_1 &= \sigma_1 u \bar{\sigma}_1 v \, (\sigma_2^2 u \bar{\sigma}_3^2 v - \sigma_3^2 u \bar{\sigma}_2^2 v), \\
\rho u_2 &= \sigma_2 u \bar{\sigma}_2 v \, (\sigma_3^2 u \bar{\sigma}_1^2 v - \sigma_1^2 u \bar{\sigma}_3^2 v), \\
\rho u_3 &= \sigma_3 u \bar{\sigma}_3 v \, (\sigma_1^2 u \bar{\sigma}_2^2 v - \sigma_2^2 u \bar{\sigma}_1^2 v).
\end{aligned}$$

These equations give us the parametric representation of the reciprocal surface. Treating the u's as point-coordinates, we find the trace on the plane $u_0 = 0$ is the pair of conics

$$\left(\frac{u_1^2}{a^2} + \frac{u_2^2}{b^2} + \frac{u_3^2}{c_2}\right)\left(\frac{u_1^2}{\bar{a}^2} + \frac{u_2^2}{\bar{b}^2} + \frac{u_3^2}{\bar{c}^2}\right) = 0,$$

with similar results for the other coordinate planes. The equation of the reciprocal surface is found to be

$$\begin{aligned}
&\frac{u_0^4}{a^2\bar{a}^2b^2\bar{b}^2c^2\bar{c}^2} + \frac{u_1^4}{a^2\bar{a}^2} + \frac{u_2^4}{b^2\bar{b}^2} + \frac{u_3^4}{c^2\bar{c}^2} + \left(\frac{1}{b^2\bar{c}^2} + \frac{1}{\bar{b}^2c^2}\right)\frac{u_0^2u_1^2}{a^2\,\bar{a}^2} \\
&\qquad + \left(\frac{1}{c^2\bar{a}^2} + \frac{1}{\bar{c}^2a^2}\right)\frac{u_0^2u_2^2}{b^2\,\bar{b}^2} + \left(\frac{1}{a^2\bar{b}^2} + \frac{1}{\bar{a}^2b^2}\right)\frac{u_0^2u_3^2}{c^2\bar{c}^2} + \left(\frac{1}{b^2\bar{c}^2} + \frac{1}{\bar{b}^2c^2}\right)u_2^2u_3^2 \\
&\qquad\qquad + \left(\frac{1}{c^2\bar{a}^2} + \frac{1}{\bar{c}^2a^2}\right)u_3^2u_1^2 + \left(\frac{1}{a^2\bar{b}^2} + \frac{1}{\bar{a}^2b^2}\right)u_1^2u_2^2 = 0 .
\end{aligned}$$

This is again a tetrahedroid surface. It may be obtained from the first by the transformation

$$\begin{aligned}
u_0 &= a\bar{a}\, b\bar{b}\, c\bar{c}\, x_0, \\
u_1 &= a\bar{a}\, x_1, \\
u_2 &= b\bar{b}\, x_2, \\
u_3 &= c\bar{c}\, x_3.
\end{aligned}$$

9. *Singular Points.* The singular points of the tetrahedroid may be found in the usual manner by obtaining the values of u and v, for which we have simultaneously $u_i = 0$ ($i = 0, 1, 2, 3$). They may be more readily found, however, by making use of the fact that singular planes reciprocate into singular points,

and vice versa. Reciprocating our sixteen singular tangent planes, we obtain at once the sixteen singular points

$$\begin{array}{rrrr} 0, & 1, & \pm 1, & \pm 1\;, \\ 1, & 0, & \pm c\bar{c}, & \pm b\bar{b}, \\ 1, & \pm c\bar{c}\,, & 0, & \pm a\bar{a}\,, \\ 1, & \pm b\bar{b}, & \pm a\bar{a}, & 0. \end{array}$$

10. It is seen that there are four singular points in each plane of reference just as there are four singular planes through each vertex of the tetrahedron of reference.

It is easily verified, moreover, that the four singular points in any plane of reference are the points where the two conics in that plane intersect. Also the line joining any two of these points in a plane of reference passes through a vertex of the tetrahedron of reference. By reciprocating, we find that the four singular planes, through any vertex of the tetrahedron of reference, intersect along lines which lie two in each of the planes of reference through that vertex. These four planes also cut the opposite plane of reference in the four common tangents of the two conics in that plane. Their points of tangency lie by twos on straight lines which also pass through a vertex of the tetrahedron of reference. These theorems, of course, are easily established directly by using the equations of the planes.

11. The singular point (0, 1, 1, 1) lies in each of the six singular planes,

$$\begin{array}{lrrr} q\,(a\bar{a}, & 0, & +1, & -1), \\ q\,(a\bar{a}, & 0, & -1, & +1), \\ q\,(b\bar{b}\,, & +1, & 0, & -1), \\ q\,(b\bar{b}\,, & -1, & 0, & +1), \\ q\,(c\bar{c}\,, & +1, & -1, & 0), \\ q\,(c\bar{c}\,, & -1, & +1, & 0) \end{array}$$

Making use of the transformations of the group of paragraph 3, we have the theorem:

Through every singular point pass six singular tangent planes.

Also, by reciprocation, or directly from the coordinates of the singular points;

There are six singular points in every singular tangent plane.

A convenient notation for the sixteen singular planes is the following:

$$
\begin{array}{rlllll}
Ia &= q\,(0, & +a\bar{a}, & +b\bar{b}, & +c\bar{c}), \\
Ib &= q\,(0, & +a\bar{a}, & +b\bar{b}, & -c\bar{c}), \\
Ic &= q\,(0, & +a\bar{a}, & -b\bar{b}, & +c\bar{c}), \\
Id &= q\,(0, & +a\bar{a}, & -b\bar{b}, & -c\bar{c}),
\end{array}
$$

$$
\begin{array}{rlllll}
IIa &= q\,(a\bar{a}, & 0, & +1, & +1); \\
IIb &= q\,(a\bar{a}, & 0, & +1, & -1); \\
IIc &= q\,(a\bar{a}, & 0, & -1, & +1); \\
IId &= q\,(a\bar{a}, & 0, & -1, & -1);
\end{array}
$$

$$
\begin{array}{rlllll}
IIIa &= q\,(b\bar{b}, & +1, & 0, & +1); \\
IIIb &= q\,(b\bar{b}, & +1, & 0, & -1); \\
IIIc &= q\,(b\bar{b}, & -1, & 0, & +1); \\
IIId &= q\,(b\bar{b}, & -1, & 0, & -1);
\end{array}
$$

$$
\begin{array}{rlllll}
IVa &= q\,(c\bar{c}, & +1, & +1, & 0); \\
IVb &= q\,(c\bar{c}, & +1, & -1, & 0); \\
IVc &= q\,(c\bar{c}, & -1, & +1, & 0); \\
IVd &= q\,(c\bar{c}, & -1, & -1, & 0);
\end{array}
$$

and similarly for the singular points:

$$
\begin{array}{rlllll}
1a &= (0, & 1, & +1, & +1); \\
1b &= (0, & 1, & +1, & -1); \\
 & \text{etc.} \\
2a &= (1, & 0, & +c\bar{c} & +b\bar{b}); \\
 & \text{etc.} \\
3a &= (1, & +c\bar{c} & 0, & a\bar{a}); \\
 & \text{etc.} \\
4a &= (1, & b\bar{b} & a\bar{a} & 0); \\
 & \text{etc.}
\end{array}
$$

With this notation we may write down the six singular points in each singular plane as follows:

$$
\begin{array}{l}
Ia:\ 2b,\ 2c,\ 3b,\ 3c,\ 4b,\ 4c; \\
Ib:\ 2a,\ 2d,\ 3a,\ 3d,\ 4b,\ 4c; \\
Ic:\ 2a,\ 2d,\ 3b,\ 3c,\ 4a,\ 4d; \\
Id:\ 2b,\ 2c,\ 3a,\ 3d,\ 4a,\ 4d;
\end{array}
$$

IIa:	3*b*,	3*d*,	4*b*,	4*d*,	1*b*,	1*c*;
IIb:	3*a*,	3*c*,	4*b*,	4*d*,	1*a*,	1*d*;
IIc:	3*b*,	3*d*,	4*a*,	4*c*,	1*a*,	1*d*;
IId:	3*a*,	3*c*,	4*a*,	4*c*,	1*b*,	1*c*;
IIIa:	4*c*,	4*d*,	1*b*,	1*d*,	2*b*,	2*d*;
IIIb:	4*c*,	4*d*,	1*a*,	1*c*,	2*a*,	2*c*;
IIIc:	4*a*,	4*b*,	1*a*,	1*c*,	2*b*,	2*d*;
IIId:	4*a*,	4*b*,	1*b*,	1*d*,	2*a*,	2*c*;
IVa:	1*c*,	1*d*,	2*c*,	2*d*,	3*c*,	3*d*;
IVb:	1*a*,	1*b*,	2*a*,	2*b*,	3*c*,	3*d*;
IVc:	1*a*,	1*b*,	2*c*,	2*d*,	3*a*,	3*b*;
IVd:	1*c*,	1*d*,	2*a*,	2*b*,	3*a*,	3*b*.

The six singular planes through each singular point are:

1*a*:	*IIb*,	*IIc*,	*IIIb*,	*IIIc*,	*IVb*,	*IVc*;
1*b*:	*IIa*,	*IId*,	*IIIa*,	*IIId*,	*IVb*,	*IVc*;
1*c*:	*IIa*,	*IId*,	*IIIb*,	*IIIc*,	*IVa*,	*IVd*;
1*d*:	*IIb*,	*IIc*,	*IIIa*,	*IIId*,	*IVa*,	*IVd*;
2*a*:	*IIIb*,	*IIId*,	*IVb*,	*IVd*,	*Ib*,	*Ic*;
2*b*:	*IIIa*,	*IIIc*,	*IVb*,	*IVd*,	*Ia*,	*Id*;
2*c*:	*IIIb*,	*IIId*,	*IVa*,	*IVc*,	*Ia*,	*Id*;
2*d*:	*IIIa*,	*IIIc*,	*IVa*,	*IVc*,	*Ib*,	*Ic*;
3*a*:	*IVc*,	*IVd*,	*Ib*,	*Id*,	*IIb*,	*IId*;
3*b*:	*IVc*,	*IVd*,	*Ia*,	*Ic*,	*IIa*,	*IIc*;
3*c*:	*IVa*,	*IVb*,	*Ia*,	*Ic*,	*IIb*,	*IId*;
3*d*:	*IVa*,	*IVb*,	*Ib*,	*Id*,	*IIa*,	*IIc*;
4*a*:	*Ic*,	*Id*,	*IIc*,	*IId*,	*IIIc*,	*IIId*;
4*b*:	*Ia*,	*Ib*,	*IIa*,	*IIb*,	*IIIc*,	*IIId*;
4*c*:	*Ia*,	*Ib*,	*IIc*,	*IId*,	*IIIa*,	*IIIb*;
4*d*:	*Ic*,	*Id*,	*IIa*,	*IIb*,	*IIIa*,	*IIIb*.

12. A number of theorems may be obtained by inspection from these two tables. In the first place, no three singular points lie in more than one singular plane, and no three singular planes pass through more than one singular point. This means that no three singular points are collinear and no three singular planes pass through one line. Again, there are two points common to any pair of planes, and two planes common to any pair of points. This means that the

120 lines joining the sixteen points are precisely the 120 lines in which the sixteen planes intersect.

The four planes $Ia\, Ib\, Ic\, Id$ all pass through the vertex $v_0 = (1, 0, 0, 0)$ of the tetrahedron of reference. Therefore their six lines of intersection do also. Now, by looking at the table, we see that the points

$2b$ and $2c$ lie in the intersection of Ia and Id;
$3b$ and $3c$ lie in the intersection of Ia and Ic;
$4b$ and $4c$ lie in the intersection of Ia and Ib.

These six points, therefore, lie by twos on three straight lines through the vertex v_0. Now, these six points all lie on the conic in Ia, and we have the theorem:

In each singular plane the six singular points are three pairs in involution, and the center of involution is a vertex of the tetrahedron of reference.

We have shown the above theorem for only one singular plane. It follows for the others by using the transformations of the group of paragraph 3.

To write the reciprocal theorem, it is necessary to see that the six planes through any singular point are tangent to a quadric cone. Consider, in fact, the six planes through $1a$. They are by the table IIb, IIc, $IIIb$, $IIIc$, IVb, IVc, or, written at length,

$$\begin{array}{llll} q\,(a\bar{a}, & 0, & 1, & -1), \\ q\,(a\bar{a}, & 0, & -1, & 1), \\ q\,(b\bar{b}, & 1, & 0, & -1), \\ q\,(b\bar{b}, & -1, & 0, & 1), \\ q\,(c\bar{c}, & 1, & -1, & 0), \\ q\,(c\bar{c}, & -1, & 1, & 0). \end{array}$$

The traces of these planes on the plane $x_1 = 0$ will give the six lines

$$\begin{array}{l} a\bar{a}\,x_0 \pm x_2 \mp x_3 = 0 \\ b\bar{b}\,x_0 \qquad\;\; \mp x_3 = 0, \\ c\bar{c}\,x_0 \pm x_2 \qquad\;\; = 0. \end{array}$$

It will suffice if we show that these lines are all tangent to the same conic. This requires the vanishing of the determinant

$$\begin{vmatrix} a^2\bar{a}^2, & 1, & 1, & a\bar{a}, & -a\bar{a}, & -1 \\ a^2\bar{a}^2, & 1, & 1, & -a\bar{a}, & a\bar{a}, & -1 \\ b^2\bar{b}^2, & 0, & 1, & 0, & -b\bar{b}, & 0 \\ b^2\bar{b}^2, & 0, & 1, & 0, & b\bar{b}, & 0 \\ c^2\bar{c}^2, & 1, & 0, & -c\bar{c}, & 0, & 0 \\ c^2\bar{c}^2, & 1, & 0, & c\bar{c}, & 0, & 0 \end{vmatrix} = 0.$$

Subtract the first row from the second, the third from the fourth, the fifth from the sixth and the determinant is seen at once to be equal to zero.

The reciprocal theorem of the one last written reads therefore:

The six singular planes through any singular point are tangent to a quadric cone and are three pairs in involution. The intersecting pairs meet in a plane of reference.

13. Consider any pair of singular points not in the same face of the tetrahedron of reference, e. g. $1c$, $4d$, we see by the table that these lie in the intersection of the two singular planes IIa and $IIIb$. The other four points in IIa are $3b$, $3d$, $4b$, and $1b$. The other four in $IIIb$ are $4c$, $1a$ $2a$, and $2c$. Now, the line joining $3b$ and $3d$ meets the line joining $1a$ and $4c$, since these two lines are in IIc. In general,

$$\overline{1b,\ 3b} \text{ meets } \overline{1a,\ 2c} \text{ in } IVc,$$
$$\overline{1b,\ 3d} \text{ meets } \overline{1a,\ 2a} \text{ in } IVb,$$
$$\overline{1b,\ 4b} \text{ meets } \overline{2a,\ 2c} \text{ in } IIId,$$
$$\overline{3b,\ 3d} \text{ meets } \overline{1a,\ 4c} \text{ in } IIc,$$
$$\overline{3b,\ 4b} \text{ meets } \overline{2c,\ 4c} \text{ in } Ia,$$
$$\overline{3d,\ 4b} \text{ meets } \overline{2a,\ 4c} \text{ in } Ib.$$

Naturally these lines intersect on the line $\overline{1c,\ 4d}$. Thus the six lines of the complete quadrilateral in IIa meet the line $\overline{1c,\ 4d}$ in the same points in which that line is met by the six lines of the complete quadrilateral in $IIIb$. A similar state of affairs exists when the two points selected lie both in the same face of the tetrahedron of reference except that here one vertex of the complete quadrilateral in each of the two singular planes will lie in the line joining the two chosen singular points, being in fact the vertex of the tetrahedron of reference lying in that line. From these considerations we have the theorem:

Every line joining two singular points meets twelve other such lines in six points which are in involution. If the two singular points lie in the same face of the tetrahedron of reference, two of these six points coincide with a vertex of the tetrahedron of reference, which is a double point of the involution.

This theorem is its own reciprocal.

UNIVERSITY OF CALIFORNIA, *August*, 1901.

On Ternary Monomial Substitution-Groups of Finite Order with Determinant ± 1.

BY ERNEST BROWN SKINNER.

INTRODUCTION.

The finite ternary linear substitution-groups generated by the two elements

$$S: \quad z_i' = z_{i+1}, \qquad T: \quad z_i' = a_i z_i,$$

$i = 1, 2, 3$, $a_1 a_2 a_3 = 1$ and $i + 1$ is taken mod 3, have been studied by Professor H. Maschke under the title "On Ternary Substitution-Groups of Finite Order which leave a Triangle Unchanged."*

Substitutions of the form

$$z_i' = a_i z_j \qquad (i, j = 1, 2, 3)$$

he has called *monomial substitutions* and the groups containing only such substitutions *monomial groups*. In what follows it is proposed to investigate all ternary monomial groups of finite order with determinant ± 1.

It is shown first, that the groups composed of multiplicative substitutions with determinant $+1$ may be generated by at most two substitutions, and conversely. The form of these independent generators is given explicitly. It is further shown that the ternary monomial groups with determinant ± 1 may be generated by at most three independent generators, one of which is of order 2, and conversely. It follows directly that the various types of groups to be studied are known. If T_1, T_2 and τ denote the generators of the ternary multiplicative group with determinant ± 1, and $S = (1, 2, 3)$, $s = (12)$, these types are found by taking every possible combination of the substitutions T_1, T_2, S, s, τ as generating operations.

In the second place, the sets of invariant forms of these groups have been

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* American Journal of Mathematics, vol. XVII, No. 2.

determined in all cases, and the full systems have been worked out in all except certain exceptional cases (see §§6 and 12 below), for which only what Professor Maschke has has called "reduced systems," have as yet been found. In these exceptional cases, while the full systems have not been found in general terms, it is shown how in any case given numerically, the forms of the full system may readily be picked out.

Finally, the orders of the various groups are given in terms of the auxiliary quantities which occur in the solution of the problems to determine the invariant systems.

CHAPTER I.

TERNARY MONOMIAL GROUPS WITH DETERMINANT $+1$.

§1.—*Definitions and Notation.*

A multiplicative ternary substitution is a monomial substitution of the form

$$z_i' = a_i z_i,$$

a_i a root of unity and $a_1 a_2 a_3 = 1$.

Such substitutions may be denoted conveniently by

$$T = (\omega_{m_1}^{k'_1}, \quad \omega_{m_2}^{k'_2}, \quad \omega_3^{k'_3}),$$

or more briefly by

$$T = (\omega_m^{k_i}),$$

where $m =$ L. C. M. of m_1, m_2, m_3, and ω_m is a primitive with root of unity.

$$i = 1, 2, 3, \qquad (2)$$

The subscript m is then the order of T and the determinant is $\omega_m^{\Sigma k} = \pm 1$.

If the determinant is $+1$

$$\Sigma k_i \equiv 0, \ (\text{mod } m). \qquad (3)$$

If it is ± 1

$$\Sigma 2k_i \equiv 0, \ (\text{mod } m). \qquad (4)$$

No two of the exponents k_i have a common divisor, which is, at the same time, a divisor of m. Two or more multiplicative substitutions T_1, T_2 .... are said to be independent if there exists no relation of the form

$$T_1^{\alpha} T_2^{\beta} \ldots\ldots = 1,$$

α, β .... not multiples of the respective orders of T_1, T_2 ....

The necessary and sufficient conditions for the equality of two multiplica-

tive substitutions $T^\alpha = (\omega_m^{k_i})^\alpha$ and $T'^\beta = (\omega_m^{k'_i})^\beta$ of order m and both of determinant $+1$, are

$$\left.\begin{array}{l} k_i\alpha - k'_i\beta \equiv 0, \\ k_j\alpha - k'_j\beta \equiv 0, \end{array}\right\} \pmod m \qquad i, j = 1, 2, 3 \quad i \neq j. \tag{5}$$

§2.—*Groups of Ternary Multiplicative Substitutions with Determinant* $+1$.

Groups of multiplicative substitutions are evidently Abelian.

Theorem I.—*The necessary and sufficient condition that two ternary multiplicative substitutions* $T_1 = (\omega_{N_1}^{k_i})$ *and* $T_2 = (\omega_{N_2}^{k'_i})$ *are independent, is that the two-rowed determinants of the matrix*

$$\left\| \begin{array}{ccc} k_1, & k_2, & k_3 \\ k'_1, & k'_2, & k'_3 \end{array} \right\|$$

are prime to $d = [N_1, N_2]$.

If one of these determinants is prime to d, the other two are also by reason of the two relations

$$\Sigma k_i \equiv \Sigma k_i \equiv 0, \pmod d.$$

Suppose first that $N_1 = N_2 = d$. The conditions for

$$T_1^\alpha T_2^\beta = 1$$

are

$$\left.\begin{array}{l} k_1\alpha + k'_1\beta \equiv 0, \\ k_2\alpha + k'_2\beta \equiv 0, \end{array}\right\} \pmod d. \tag{6}$$

If $(k_1 k'_2) = \Delta$, then

$$\left.\begin{array}{l} \Delta\alpha \equiv 0, \\ \Delta\beta \equiv 0, \end{array}\right\} \pmod d.$$

If $[\Delta, d] \neq 1$, there exists a solution of (6) such that α and β are both less than d. If $[\Delta, d] = 1$, there exists no solution of (6) except $\alpha \equiv \beta \equiv 0 \pmod d$. The condition is therefore necessary and sufficient when $N_1 = N_2$.

If $N_1 \neq N_2$, let $N_1 = r_1 d$, $N_2 = r_2 d$, $T_1^{r_1} = (\omega_d^{\bar{k}_i})$ and $T_2^{r_2} = (\omega_d^{\bar{k}'_i})$; then $k_i \equiv \bar{k}_i \pmod d$ and $k'_i = \bar{k}'_i \pmod d$. The condition that $T_1^{r_1}$ and $T_2^{r_2}$ are independent is that $\Delta = (\bar{k}_1 \bar{k}'_2)$ is relatively prime to d. But

$$\Delta = \bar{\Delta} + d\,(\text{int. fcn. } \bar{k}_i, \bar{k}_i).$$

If, therefore, $[\bar{\Delta}, d] = 1$, $[\Delta, d] = 1$, and conversely. Q. E. D.

Corollary. *If N_2 is a divisor of N_1, the condition that* $T_1 = (\omega_{N_1}^{k_i})$ *and* $T_2 = (\omega_{N_2}^{k'_i})$ *are independent is*

$$[\Delta, N_2] = 1.$$

THEOREM II.—*Every group of ternary multiplicative substitutions with determinant $+1$ may be generated by at most two independent generators, and conversely, every Abelian group that can be generated by two generators is holoedrically isomorphic with a group of ternary multiplicative substitutions.*

First, let G be a group of order p^n, p a prime, and let

$$T_1 = (\omega_{p^{n_1}}^{k_i}) \qquad n_1 \ngtr n$$

be a substitution of maximum order in the group. If G is exhausted by the powers of T_1, the group is cyclic. If G contains yet other operations, let

$$T_2 = (\omega_{p^{n_2}}^{k'_i})$$

be a substitution of maximum order among the remaining elements which are independent of T_1. The group $\{T_1, T_2\}$ will then contain every ternary multiplicative substitution whose order is a divisor of p^{n_2}. For the conditions that

$$(T_1^{p^{n_1}-p^n})^{\alpha}\, T_2^{\beta} = T = (\omega_{p^{n_2}}^{m_i}),$$

T arbitrary and of order p^{n_2} or less are

$$\left.\begin{aligned} \bar{k}_1\alpha + k'_1\beta &\equiv m_1, \\ \bar{k}_2\alpha + k'_2\beta &\equiv m_2, \end{aligned}\right\} \pmod{p^{n_2}}. \tag{7}$$

The congruences (7) have a solution whatever m_1 and m_2 may be since, by Theorem I, $[(k_1 k'_2), p^{n_2}] = 1$. There cannot then be a third independent generator.

The converse is easily shown to be true. For, let Γ be any Abelian group of order p^n with two independent generators. Its "Weber invariants"* are then p^n and p^{n_2} where $n_1 + n_2 = n$. Let

$$T_1 = (\omega_{p^{n_1}}^{A_i})$$

be any substitution of order p^{n_1}. It is possible to find a set of numbers B_i which, together with A_i, satisfy the conditions of Theorem I, and for which $\Sigma B_i \equiv 0 \pmod{p^{n_2}}$. Moreover, the notation may be chosen so that $n_1 \geqq n_2$.

To prove the theorem in the general case, let G_N be a ternary multiplicative group of order

$$N = p_1^{a_1} p_2^{a_2} \ldots . p_\lambda^{a_\lambda}, \qquad p_i \text{ a prime.}$$

* Weber, "Algebra," vol. II, §12.

Let $T_1^{(i)}$ and $T_2^{(i)}$ be the generators of the subgroup of order $p_i^{a_i}$ which exists in G_{N_1}. Further, let

$$G_{N_1} = \{ T_1^{(1)}, T_1^{(2)}, \ldots, T_1^{(\lambda)} \}$$

and

$$G_{N_2} = \{ T_2^{(1)}, T_2^{(2)}, \ldots, T_2^{(\lambda)} \},$$

whose orders are

$$N_1 = \prod_1^\lambda p_i^{n_1^{(i)}} \text{ and } N_2 = \prod_1^\lambda p_i^{n^{(i)}} \tag{8}$$

respectively. Moreover,

$$N_1 N_2 = N. \tag{9}$$

The orders of $T_1^{(i)}$ are relatively prime. Hence,

$$G_{N_1} = \{ T_1 \} \text{ where } T_1 = T_1^{(1)} . T_1^{(2)} \ldots . T_1^{(\lambda)}.^*$$

Similarly,

$$G_{N_2} = \{ T_2 \} \text{ where } T_2 = T_2^{(1)} T_2^{(2)} T_2^{(3)} \ldots . T_2^{(\lambda)}.$$

If $n_1^{(i)} \geqq n_2^{(i)}$ for every i, $[N_1, N_2] = N_2$.

T_1 and T_2 are independent, for suppose $T_1^m = T_2^n$, m and n any positive integers; then

$$T_1^{\frac{mN_1}{p_i^{n_i^{(i)}}}} = T_2^{\frac{nN_1}{p^{n^{(i)}}}}$$

reduces to

$$T_1^{(i)\frac{mN_1}{p^{n_1^{(i)}}}} = T_2^{(i)\frac{nN_1}{p_i^{n_1^{(i)}}}},$$

Consequently m contains $p_i^{n_1^{(i)}}$ and n contains $p_i^{n_2^{(i)}}$. Therefore, m contains N_1 and n contains N_2. Conversely, let Γ_N be any Abelian group of order N which can be generated by two generators so that

$$\Gamma_N = \{ \theta_1, \theta_2 \}.$$

Among the Weber invariants of Γ_N, not more than two powers of any prime p_i can be found, viz. one which is a divisor of the order of θ_1 and the other a divisor of the order θ_2. But the Weber invariants of the most general group of ternary multiplicative substitutions are

$$p_1^{n_1^{(1)}}, p_1^{n_2^{(1)}}, p_2^{n_1^{(2)}}, p_2^{n_2^{(2)}} \ldots . p_\lambda^{n_1^{(\lambda)}}, p_\lambda^{n_2^{(\lambda)}},$$

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* Weber, "Algebra," vol. II, §12.

and among the possible values of $n_1^{(i)}$ and $n_2^{(i)}$ may be found every bipartite partition of a_i, i. e. among the groups G_N will be found groups isomorphic with all possible Abelian groups that may be generated by two generators. Q. E. D.

§3.—*The Forms which remain Invariant with respect to the Substitutions of the Group* $\{T_1, T_2\}$.

Let $T_1 = (\omega_{N_1}^{k_i})$ and $T_2 = (\omega_{N_2}^{k_i'})$ be the generators of the group $\{T_1, T_2\}$ so chosen that $[N_1, N_2] = N_2$.

The invariant forms of $\{T_1, T_2\}$ are rational integral functions of monomial forms of the three types:

$$\left.\begin{array}{lll} \text{I.} & z_i^\alpha, & i=1,2,3, \\ \text{II.} & z_i^\alpha z_j^\beta, & i,j=1,2,3, \quad i \neq j, \\ \text{III.} & (z_1z_2z_3)^\alpha. & \end{array}\right\} \tag{10}$$

The conditions for the invariance of z_i^α are

$$\left.\begin{array}{l} k_i\alpha \equiv 0 \pmod{N_1}, \\ k_i'\alpha \equiv 0 \pmod{N_2}. \end{array}\right\} \tag{11}$$

By Theorem I, $[(k_ik_j'), N_2] = 1$. Hence,

$$\alpha \equiv 0 \pmod{N_2}.$$

Let $\alpha = \alpha_1 N_2$,

and let $N_1 = N_2 . \bar{N}_1$. (12)

Also let $[k_i, \bar{N}] = q_i$; then

$$\alpha_1 \equiv 0 \mod \frac{\bar{N}}{q_i}. \tag{13}$$

The least value of α is therefore

$$\alpha = \alpha_1 . \frac{\bar{N}}{q_i} = \frac{N_1}{q_i}. \tag{14}$$

The forms of type I are then given by

$$z_i^{\lambda \frac{N_1}{q_i}}, \qquad \lambda \text{ any positive integer.}$$

The conditions for the invariance of the form $z_i^\alpha z_j^\beta$ are

$$\left.\begin{array}{l} k_i\alpha + k_j\beta \equiv 0 \pmod{N_1}, \\ k_i'\alpha + k_j'\beta \equiv 0 \pmod{N_2}. \end{array}\right\} \tag{15}$$

Let $k_i k_j' - k_i' k_j \equiv \Delta$; then, since N_1 contains N_2,

$$\Delta\alpha \equiv 0 \pmod{N_2},$$
$$\Delta\beta \equiv 0 \pmod{N_2}.$$

But $[\Delta, N_2] = 1$, by Theorem I, so that

$$\alpha \equiv \beta \equiv 0 \pmod{N_2}.$$

The congruences (15) then reduce to the single congruence

$$k_i\alpha_1 + k_j\beta_1 \equiv 0 \pmod{\bar{N}}. \tag{16}$$

For $N_1 = N_2$ and consequently $\bar{N} = 1$, (16) has no meaning, but in this case the solution of (15) is

$$\alpha \equiv \beta \equiv 0 \pmod{N_1}.$$

For $\bar{N} > 1$, let $k_i = q_i k_i$, $i = 1, 2, 3$, where $q_i = [k_i, \bar{N}]$, and

$$\bar{N} = q_i q_j \bar{N}'. \tag{17}$$

From (16) it follows that α_1 contains q_j and β_1 contains q_i, so that (16) reduces to

$$k_i\alpha_2 + k_j\beta_2 \equiv 0 \pmod{\bar{N}'}, \tag{18}$$

where $\quad \alpha_1 = q_j\alpha_2, \quad \beta_1 = q_i\beta_2.$

To solve (18), put

$$\alpha_2 = n \pmod{\bar{N}'},$$

then
$$k_i n + k_j\beta_2 \equiv 0 \bmod \bar{N}'. \tag{19}$$

Let v be the least positive solution of the congruence

$$vk_j \equiv -k_i \pmod{\bar{N}'}. \tag{20}$$

From (19) and (20) it follows that

$$\beta_2 = nv \pmod{\bar{N}'}.$$

Let v_n be defined by

$$v_n \equiv nv \pmod{\bar{N}'}.$$

The general solution of (18) is then

$$\left.\begin{aligned} \alpha_2 &= n + \lambda\bar{N}', \\ \beta_2 &= v_n + \mu\bar{N}', \end{aligned}\right\} \lambda, \mu \text{ integers,}$$

whence the solution of (15) is

$$\left.\begin{aligned} \alpha &= N_2 q_j (n + \lambda\bar{N}'), \\ \beta &= N_2 q_i (v_n + \mu\bar{N}'). \end{aligned}\right\} \tag{21}$$

We have then the proposition:

THEOREM III.—*The invariant forms of the group* $\{T_1, T_2\}$ *are rationl integral functions of the following forms:*

$$\left.\begin{array}{lll} \text{I.} & z_i^{\frac{N_1}{q_i}} & i = 1, 2, 3. \\ \text{II.} & (z_i^{nq_j} z_j^{v_n q_i})^{N_2} & i, j = 1, 2, 3. \\ \text{III.} & z_1 z_2 z_3, & \end{array}\right\} \tag{22}$$

where N_1 *and* N_2 *are the respective orders of* T_1 *and* T_2, $q_i = [k_i, N_1 \div N_2]$, n *is a positive integer* $< \dfrac{N_1}{N_2 q_i q_j}$ *and* v_n *is defined by the congruences*

$$k_j v + k_i \equiv 0 \left(\operatorname{mod} \frac{N_1}{N_2 q_i q_j}\right), \qquad v_n \equiv nv \left(\operatorname{mod} \frac{N_1}{N_2 q_i q_j}\right).$$

The full system* is easily found. The forms $z_i^{\frac{N_1}{q_i}}$ $i = 1, 2, 3$ and $z_1 z_2 z_3$ belong to the full system. It remains only to examine the forms

$$z_i^{nq_j N_2} z_j^{v_n q_i N_2} \tag{23}$$

obtained by allowing n to run through the set of values 1, 2, ... $\bar{N}'$—1, where $\bar{N}'$ is defined by (17).

Recurring to the definition of $\bar{N}'$, it is seen that there are

$$\frac{\bar{N}}{q_1 q_2 q_3}(q_1 + q_2 + q_3) - 3$$

forms of the type (23). These forms, together with the four forms $z_i^{\frac{N_1}{q_i}}$ $i = 1, 2, 3$ and $z_1 z_2 z_3$ include the full system and, in some cases, coincide with it. This system of $\dfrac{\bar{N}}{q_1 q_2 q_3}(q_1 + q_2 + q_3) + 1$ forms is called the "reduced system."†

Suppose it be possible that for some partition of n for $n < \bar{N}'$,

$$\left.\begin{array}{l} n_1 + n_2 + \cdots + n_\lambda = n \\ v_{n_1} + v_{n_2} + \cdots + v_{n_\lambda} = v_n \end{array}\right\}, \quad n < \bar{N}' \tag{24}$$

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* The full system is defined to be a set of forms, the fewest possible in number, in terms of which every other form of the system is rationally expressible.

† See Professor Maschke's paper where the expression is used in a slightly different though strictly analogous sense.

hold simultaneously. It is evident that all forms $z_i^{nq_jN_2} z_j^{v_n q_i N_2}$, for which relations similar to (24) hold simultaneously, do not belong to the full system. It follows that the full system of the group $\{T_1, T_2\}$ consists of the forms $z_i^{\frac{N_1}{q_i}}$, $i = 1, 2, 3$, $z_1 z_2 z_3$ and those forms $z_i^{nq_jN_2} z_j^{v_n q_i N_2}$, for which the relations (24) are not simultaneously true.

§4.—*The Invariant Forms of the Group* $\{T_1, T_2, S\}$.

The group $\{T_1, T_2, S\}$, where S denotes the cyclic substitution $(z_1 z_2 z_3)$, is the most general ternary monomial group with determinant $+1$.

Since every invariant form is unchanged by S, the following types are admissible:

$$\left.\begin{array}{rl} \text{I.} & z_1^\alpha + z_2^\alpha + z_3^\alpha, \\ \text{II.} & z_1^\alpha z_2^\beta + z_2^\alpha z_3^\beta + z_3^\alpha z_1^\beta, \\ \text{III.} & z_1^\alpha z_2^\beta z_3^\gamma + z_2^\alpha z_3^\beta z_1^\gamma + z_3^\alpha z_1^\beta z_2^\gamma. \end{array}\right\} \tag{25}$$

If ρ be the least of the three integers α, β, γ in type III, the form is divisible by the invariant $(z_1 z_2 z_3)^\rho$, while the remaining factor is either of type I or of type II.

For I, we may write (z_1^α) and for II $(z_1^\alpha z_2^\beta)$. It follows directly that the forms which remain invariant with respect to the group $\{T_1, T_2, S\}$ are rational integral functions of $z_1 z_2 z_3$ and of forms of the types (z_1^α) and $(z_1^\alpha z_2^\beta)$.

The forms (z_1^α) go into $(\omega_{N_1}^{k_1 \alpha} z_1^\alpha)$ by the substitution T_1, whence it follows that $\alpha \equiv 0 \pmod{N_1}$ is a necessary condition. This condition is also sufficient, since N_1 contains N_2. The invariant forms of the type (z_1^α) are then all given by $(z_1^{\lambda N_1})$, where λ is any positive integer.

The conditions that a term $z_i^\alpha z_j^\beta$ of $(z_1^\alpha z_2^\beta)$ shall be invariant are

$$\left.\begin{array}{l} k_i\alpha + k_j\beta \equiv 0 \pmod{N_1}, \\ k_i\alpha + k_j\beta \equiv 0 \pmod{N_2}, \end{array}\right\} \tag{26}$$

But these congruences are identical with (15) and consequently reduce to the single congruence

$$k_i\alpha_1 + k_j\beta_1 \equiv 0 \pmod{\bar{N}},$$

with notation the same as in §3. Giving to i and j all possible values, and remembering that $\Sigma k_i \equiv 0 \bmod \bar{N}$, we find

$$\left.\begin{array}{l} k_1\alpha_1 + k_2\beta_1 \equiv 0 \\ k_2\alpha_1 - (k_1 + k_2)\beta_1 \equiv 0 \end{array}\right\} \bmod \bar{N}. \tag{27}$$

Let $c=[k_1, k_2]$, and as before $q_i=[k_i, \bar{N}]$, then $[c, q_i]=1$ by reason of $\Sigma k_i \equiv 0 \pmod{N}$. Let $k_1=c\varkappa_1$, $k_2=c\varkappa_2$, then the congruences (26) show that

$$\alpha \equiv \beta \equiv 0 \pmod{q_1 q_2 q_3}.$$

Let
$$q_1 q_2 q_3 = Q, \quad \bar{N}=QR, \quad \alpha_1 = Q\alpha_2, \quad \beta_1 = Q_2\beta_2. \tag{28}$$

When the factors q_1, q_2, q_3 and c are divided out, (26) takes the form

$$\left.\begin{aligned} \varkappa_1\alpha_2 + \varkappa_2\beta_2 &\equiv 0 \\ \varkappa_2\alpha_2 - (\varkappa_1+\varkappa_2)\,\beta_2 &\equiv 0 \end{aligned}\right\} \pmod{R}. \tag{29}$$

The coefficients of (28) are relatively prime to R. Let

$$\Delta = \varkappa_1^2 + \varkappa_1\varkappa_2 + \varkappa_2^2, \quad t=[\Delta, R], \quad \Delta = st, \quad R=rt. \tag{30}$$

It follows that α and β contain the factor r and the congruences (29) reduce to

$$\left.\begin{aligned} k_1\alpha_3 + k_2\beta_3 &\equiv 0\,, \\ k_2\alpha_3 - (k_1+k_2)\,\beta_3 &\equiv 0\,, \end{aligned}\right\} \pmod{t}, \tag{31}$$

where
$$\alpha_2 = r\alpha_3, \quad \beta_2 = r\beta_3. \tag{32}$$

The first congruence of (31) is identical in form with (18). Its solution is therefore

$$\left.\begin{aligned} \alpha_3 &= n + \lambda t, \\ \beta_3 &= v_n + \mu t, \\ v\varkappa_2 + \varkappa_1 &\equiv 0 \pmod{t}, \\ v_n &\equiv nv \pmod{t}. \end{aligned}\right\} \tag{33}$$

It is easy to show that this solution satisfies the second of (31). It is therefore the general solution of (31). Let

$$\vartheta = N_2\, Qr, \tag{34}$$

then, by reason of (27), (31) and (32), the solution of (26) is

$$\left.\begin{aligned} \alpha &= \vartheta\,(n + \lambda t), \\ \beta &= \vartheta\,(v_n + \mu t), \\ vk_2 + k_1 &\equiv 0 \pmod{t}, \\ v_n &\equiv nv \pmod{t}, \end{aligned}\right\} \tag{35}$$

where $n = 0, 1, 2, \ldots . \; t-1$.

If the solution had been found by making $\beta_3 \equiv n \pmod t$, it would have taken the form

$$\left.\begin{aligned} \alpha &= \vartheta(w_n + \lambda' t), \\ \beta &= \vartheta(n + \mu' t), \\ w x_1 + x_2 &\equiv 0 \pmod t, \\ w_n &\equiv nw \pmod t. \end{aligned}\right\} \tag{36}$$

The results obtained in the present section may be summed up as follows:

THEOREM IV.—*The invariant forms of the group* $\{T_1, T_2, S\}$ *are rational integral functions of the following forms;*

$$\left.\begin{array}{ll} \text{I.} & (z_i^{N_1\lambda}),\ \lambda \text{ a positive integer.} \\ \text{II.} & (z_1^{\vartheta(n+\lambda t)} z_2^{\vartheta(v_n+\mu t)}),\ \lambda, \mu \text{ positive integers.} \\ \text{III.} & (z_1 z_2 z_3), \end{array}\right\} \tag{37}$$

where the following definitions are to be observed: N_1 *and* N_2 *are the orders of* T_1 and T_2, $N_1 = N_2 \bar{N}$, $q_i = [k_i, \bar{N}]$, $\bar{N} = QR$,

$$t = [R,\ x_1^2 + x_1 x_2 + x_2^2], \quad R = rt, \quad \vartheta = NrQ.^*$$

§5.—*The Quantities* v, w *and* t.

In the paper referred to above, Professor Maschke has given some relations between v, w and t which will be found useful in later investigations. The proofs, which are simple, will be found in Professor Maschke's paper.

1). $vw \equiv 1 \pmod t$. (38)

2). $v + w = t + 1$. (39)

3). v and w satisfy the congruence

$$x^2 - x + 1 \equiv 0 \pmod t. \tag{40}$$

4). v and w are always distinct except for $t = 3$, in which case $v = w = 2$.

5). t as a number of the form

$$p_1^{\lambda_1} p_2^{\lambda_2} \dots \quad \text{or} \quad 3p_1^{\lambda_1} p_2^{\lambda_2}, \tag{41}$$

where p_i is a prime number of the form $3h + 1$. To these properties two others may be added.

---

* The solution of the congruences (26) occurs in a slightly different form in Professor Maschke's paper.

6). The solution of the congruence (40) is possible for those and only those numbers $t = 3^{\delta} p_1^{\lambda_1} p_2^{\lambda_2} \ldots$, $\delta = 0$ or 1.

For, since $[t, 4] = 1$, (40) is equivalent to

$$4x^2 - 4x + 4 \equiv 0 \pmod{t}$$

or

$$(2x+1)^2 + 3 \equiv 0 \pmod{t}.$$

Making $y = 2x + 1$, we have

$$y^2 + 3 \equiv 0 \pmod{t}.$$

If D be any number, the divisors of $y^2 - D$ are identical with the divisors of $z^2 - Du^2$.* The divisors of $z^2 + 3u^2$ are those and only those prime numbers of the form $3h + 1$.† From the existence of these solutions we may infer the existence of solutions of the form

$$t = 3^{\delta} p_1^{\lambda_1} p_2^{\lambda_2} \ldots, \text{‡}$$
$$\delta = 0 \text{ or } 1.$$

7). Finally, for $t \equiv 0 \pmod{3}$,

$$[v - w, t] = 3$$

and for $t \not\equiv 0 \bmod 3$

$$[v - w, t] = 1.$$

For, from (38) and (39), one finds

$$(v - w)^2 + 3 \equiv 0 \pmod{t}. \tag{42}$$

6.—*The Full System for the Group* $\{T_1, T_2, S\}$.

If one makes in (37) the substitution $z_i^{\vartheta} = y_i$, the invariant forms of the group become rational integral functions of the forms

$$\text{I.}\ (y_1^{kt}), \qquad \text{II.}\ (y_1^{n+\lambda t} y_2^{v_n + \mu t}), \qquad \text{III.}\ \sqrt[\vartheta]{y_1 y_2 y_3}. \tag{43}$$

The cases $t = 1$ and $t > 1$ are treated separately. For $t = 1$ the congruences (29) are satisfied by any positive integral values for α_3 and β_3. Since $\alpha = \vartheta\alpha_3$ and $\beta = \vartheta\beta_3$, the forms $(y_1^{\alpha} y_2^{\beta})$ are invariant for every positive integral α and β. We have

$$(y_1^{\alpha} y_2^{\beta}) + (y_1^{\beta} y_2^{\alpha}) = S_1, \text{ a symmetric function;}$$
$$(y_1^{\alpha} y_2^{\beta}) - (y_1^{\beta} y_2^{\alpha}) = S_2 \Delta, \text{ an alternating function,}$$

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* Dirchlet-Dedekind, "Zahlentheorie," §52, ed. 1894. † Ibid. §70. ‡ Ibid. §35.

where S_2 is a symmetric function and Δ is the discriminant of $y_1 y_2 y_3$. We have then

$$(y_1^\alpha y_2^\beta) = \frac{S_1 + S_2\Delta}{2},$$

from which it follows immediately that the full system consists of the four forms

$$(y_1), \quad (y_1 y_2), \quad \sqrt[3]{y_1 y_2 y_3} \text{ and } \Delta. \tag{44}$$

Between these four forms there exists the relation

$$\Delta^2 = 18\,(y_1)\cdot(y_1 y_2)\cdot(y_1 y_2 y_3) - 4\,(y_1)^3\cdot y_1 y_2 y_3 - 4\,(y_1 y_2)^3 + (y_1 y_2)^2\cdot(y_1)^2 - 27\,(y_1 y_2 y_3)^3. \tag{45}$$

Case II. $t > 3$.

If ψ_n be defined by $\psi_n = (y_1^n y_2^{v_n})$, there are $t-1$ forms which are obtained by allowing n to run from 1 to $t-1$. Professor Maschke has proven the following theorem:

Theorem V.—*If* $t > 1$, *every invariant form of the system* (43) *is expressible rationally in terms of the "reduced system," consisting of the* $t+1$ *forms* (y_1^t), $\sqrt[3]{y_1 y_2 y_3}$ *and* Ψ_n, $\quad n = 1, 2, 3, \ldots\, t-1$.*

A general expression for the full system of such a reduced system has not been found, but in any given case the forms of the full system may be picked out by means of the following theorem:

Theorem VI.—*The full system of the group* $\{T_1, T_2, S\}$, $t > 1$, *consists of the forms* (y_1^t), $\sqrt[3]{y_1 y_2 y_3}$ *and those forms* ψ_n *for which the relations*

$$n_1 + n_2 + \ldots = n, \quad v_{n_1} + v_{n_2} + \ldots = v_n \qquad (n, n_i \leqq t-1) \tag{46}$$

are not both true.

Let ψ_n be a ψ of minimum order in the set of ψ's for which the property in question is true. Then

$$\psi_{n_1}\psi_{n_2}\ldots - \psi_n + (\sqrt[3]{y_1 y_2 y_3})^\mu\,\Psi, \qquad \mu > 0, \tag{47}$$

where Ψ is a function of the forms ψ. According to hypothesis, the form Ψ cannot contain any ψ for which the property in question is true. Let $\psi_{n'}$ be another function of the set and of the next higher order, then

$$\psi_{n'} = \psi_{n'_1}\,\psi_{n_2}\ldots - A^{\mu'}\Psi',$$

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* Maschke, loc. cit., p. 179.

Ψ cannot contain $\psi_{n'}$, and if it contains functions of the set ψ_n of lower degree they may be eliminated by a relation having the form (47). This process may be carried on so long as any of the ψ's having the property (47) remain. It is easy to show that it cannot be carried farther. Hence the theorem is true.

Cor. For $t > 3$ *the number of forms* ψ_n *belonging to the full system cannot exceed* $\frac{1}{2}(t-1)$ *when* $t \not\equiv 0$ *mod* 3 *or* $\frac{1}{2}(t-5)$ *when* $t \equiv 0$ (*mod* 3).

w may be taken less than v. If k be any positive integer such that

$$w + k \overline{\gtrless} t - 1,$$

then by (38),
$$v_{w+k} = v_w + v_k.$$

Consequently the number of ψ's belonging to the full system is less than w. But by (39),

$$v + w = t + 1.$$

Therefore, for $t > 3$,
$$w \overline{\gtrless} \tfrac{1}{2}(t-1).$$

For $t \equiv 0$ (mod 3) we have, by §5, 7),

$$v = w + 3m, \quad m \text{ a positive integer,}$$

and
$$2w = t + 1 - 3m.$$

The least value of m is 2, whence,

$$w \overline{\gtrless} \tfrac{1}{2}(t-5).$$

CHAPTER II.

TERNARY MONOMIAL GROUPS WITH DETERMINANT ± 1.

§7.—*Groups of Ternary Multiplicative Substitutions with Determinant* ± 1.

In any ternary multiplicative group with determinant ± 1, those substitutions with determinant $+1$ form a subgroup with index 2. Let G_{2r} be a group with determinant ± 1 and G_r the subgroup with determinant $+1$. If the Weber invariants of G_r are

$$p_1^{n_1^{(1)}}, \quad p_1^{n_2^{(1)}}, \quad p_2^{n_1^{(2)}}, \quad p_2^{n_2^{(2)}} \ldots\ldots,$$

the Weber invariants of G_{2r} are

$$2, \quad p_1^{n_1^{(1)}}, \quad p_1^{n_2^{(1)}}, \quad p_2^{n_1^{(2)}}, \quad p_2^{n_2^{(2)}} \ldots\ldots \qquad (48)$$

But if τ denote one of the substitutions

$$\tau_1 = (-1,\ 1,\ 1),\quad \tau_2 = (1,\ -1,\ 1),\quad \tau_3 = (1,\ 1,\ -1),\quad \tau_4 = (-1,\ -1,\ -1), \tag{49}$$

the Weber invariants of the group $\{T_1,\ T_2,\ \tau\}$ are precisely the numbers (48) We have then the proposition:

THEOREM VII.—*The most general ternary multiplicative group with determinant* ± 1 *is the group* $\{T_1,\ T_2,\ \tau\}$, *where* τ *is a substitution of order* 2 *and determinant* -1. *The Weber invariants of the group are*

$$2,\quad p_1^{n_1^{(1)}},\quad p_1^{n_1^{(2)}},\quad p_2^{n_2^{(1)}},\quad p_2^{n_2^{(2)}} \ldots\ldots,\ p_i \neq p_k. \tag{50}$$

Cor. If all the numbers p are odd primes, the group $\{T_1,\ T_2,\ \tau\}$ *is holoedrically isomorphic with one of the groups* $\{T_1,\ T_2\}$.

§8.—*The Invariant Forms of the Group* $\{T_1,\ T_2,\ \tau\}$.

By reason of the Corollary to Theorem VII only those groups $\{T_1,\ T_2,\ \tau\}$, for which N_1 and N_2 are both even, need be investigated. Let $p_1 = 2$. The group $\{T_1,\ T_2,\ \tau\}$ will then contain two independent substitutions T_1' and T_2' of orders $2^{n^{(1)}}$, $2^{n_1^{(2)}}$ respectively. Therefore, $T_1'^{2n_1^{(1)}-1}$ and $T_2'^{2n_1^{(2)}-1}$ are two of the substitutions

$$\sigma_1 = (1,\ -1,\ -1),\quad \sigma_2 = (-1,\ 1,\ -1),\quad \sigma_3 = (-1,\ -1,\ 1). \tag{51}$$

But $\sigma_i\sigma_j = \sigma_k$, $i, j, k = 1,\ 2,\ 3$ in some order, and

$$\tau_i\sigma_i = \tau_4,\quad \tau_4\sigma_i = \tau_i,\qquad i = 1,\ 2,\ 3,\ \ldots.$$

The group $\{T_1,\ T_2,\ \tau\}$ contains $\tau_1,\ \tau_2,\ \tau_3,\ \tau_4$ and the invariant forms are functions of $z_1^2,\ z_2^2,\ z_3^2$.

Let $N_1 = 2^{\lambda_1}Q_1$ and $N_2 = 2^{\lambda_2}Q_2$, $\lambda_2 \neq 0$ and $\overline{\gtrless} \lambda_1$, and $Q_1 \div Q_2 = \bar{Q}$, then $\bar{N} = 2^{\lambda_1-\lambda_2}\bar{Q}$ and $q_i = [k_i,\ \bar{N}]$ contains at most $2^{\lambda_1-\lambda_2}$. Therefore, $\dfrac{N_1}{q_i}$ contains 2^{λ_2} at least. It follows that all the forms (22) are invariant with respect to $\{T_1,\ T_2,\ \tau\}$ except $z_1,\ z_2,\ z_3$.

THEOREM VIII.—*The invariant forms of the group* $\{T_1,\ T_2,\ \tau\}$ *are rational integral functions of the forms*

$$\left.\begin{array}{lll} \text{I.} & z_i^{\frac{N_1}{q_i}} & i = 1,\ 2,\ 3. \\ \text{II.} & (z_i^{nq_j} z_j^{v_n q_i})^{N_2} & i, j = 1,\ 2,\ 3,\quad i \neq j. \\ \text{III.} & (z_1 z_2 z_3)^2. & \end{array}\right\} \tag{52}$$

The problem of finding the full system is the same as that in finding the full system for the group $\{T_1, T_2\}$ as may be seen by making $z_i^2 = y_i$, $i = 1, 2, 3$.

§9.—*The Invariant Forms of the Groups* $\{T_1, T_2, S, \tau\}$.

The invariant forms of the group $\{T_1, T_2, S\}$ were found to be

$$\left.\begin{array}{ll} & (z_1^{N_1 k}), \quad (z_1^{(n+\lambda t)\vartheta} z_2^{(v_n+\mu t)\vartheta}), \quad (z_1 z_2 z_3)^\nu \\ \text{or} & (y_1^{kt}), \quad (y_1^{n+\lambda t} y_2^{v_n+\mu t}), \quad \sqrt[\vartheta]{(y_1 y_2 y_3)^\nu}. \end{array}\right\} \tag{37}$$

To abbreviate the notation still further, let

$$H'_\kappa = (y_1^{\kappa t}), \quad \psi_{n,\lambda,\mu} = (y_1^{n+\lambda t} y_2^{v_n+\lambda t}) \text{ and } A = \sqrt[\vartheta]{y_1 y_2 y_3}. \tag{53}$$

$\psi_{n,\sigma,0}$ is then simply ψ_n, and the set of forms (37) takes the form

$$H_\kappa, \quad \psi_{n,\lambda,\mu}, \quad A^\nu, \tag{37a}$$

For the case N_1 even, since ϑ is even when N_1 is even, the set of forms is

$$H_\kappa, \quad \psi_{n,\lambda,\mu}, \quad A^{2\nu}. \tag{54}$$

For N_1 odd there are two subcases, viz. α) $\tau = \tau_4$; β) $\tau \neq \tau_4$.

α) $\tau = \tau_4$. The invariant forms are

$$\left.\begin{array}{l} H_{2\kappa}, \quad \psi_{n,\lambda,\mu}, \quad n + v_n + \lambda + \mu \equiv 0 \pmod 2, \\ A^{2\nu}, \quad H_\kappa A^\rho, \quad \kappa \equiv \rho \equiv 1 \pmod 2, \\ \psi_{n,\lambda,\mu} A^\rho, \quad n + v_n + \lambda + \mu \equiv \rho \equiv 1 \pmod 2, \end{array}\right\} \tag{55}$$

β) $\tau \neq \tau_4$. Since

$$S^{-1}\tau_1 S = \tau_2, \quad S^{-1}\tau_2 S = \tau_3, \quad S^{-1}\tau_3 S = \tau_1, \quad \tau_1\tau_2\tau_3 = \tau_4,$$

the invariant forms must be functions of z_1^2, z_2^2, z_3^2. They are therefore

$$H_\kappa, \quad \psi_{n,\lambda,\mu}, \quad A^\nu, \quad \kappa \equiv n + \lambda \equiv v_n + \mu \equiv \nu \equiv 0 \pmod 2. \tag{56}$$

It remains to find the full systems. For the case N_1 even, we know that H_κ and $\psi_{n,\lambda,\mu}$ are expressible rationally in terms of H_1, ψ_n and A^ϑ. It follows that the reduced system consists of the $t + 1$ forms,

$$H_1, \quad \psi_n, \quad n = 1 \ldots\ldots t - 1 \text{ and } A^2. \tag{57}$$

For the case N_1 odd and $\tau = \tau_4$, we note that by Theorem V the set of forms (55) will be included in the system consisting of the following:

$$\left.\begin{array}{l} \text{1). The even forms } \psi_n. \\ \text{2). The products } \psi_{n_1} . \psi_{n_2} \text{ of two odd forms.} \\ \text{3). The products } H_1 . \psi_n \text{ where } \psi_n \text{ is an odd form.} \\ \text{4). The products } A . \psi_n,\ \psi_n \text{ odd.} \end{array}\right\} \qquad (58)$$

To show that the form H_1^2 may he replaced by H_2, or vice versa, we have

$$\begin{aligned} H_1^2 &= H_2 + 2\,(y_1^t y_2^t) \\ &= H_2 + \text{Rat. fcn.}\ (\psi_n,\ A^\vartheta,\ A^\vartheta . H).^* \end{aligned}$$

That the reduction cannot be carried further in the general case is apparent from the case $t = 3$, since, for $t = 3$, all the forms (57) are found in the full system.† In most cases, however, it will happen that the system (57) admits of further reduction.

For the case N_1 odd and $\tau \neq \tau_4$ the invariant forms are the set (56) and these may be expressed in the form

$$H_\kappa(y^2), \quad \psi_{n,\lambda,\mu}(y^2), \quad A^\nu(y^2).$$

We find, for the even values of n, λ is even and

$$n + \lambda t = 2\,(n' + \lambda' t) \qquad n' = 1,\ 2,\ 3 \ldots\ldots \frac{t-1}{2}.$$

and for odd values of n, λ is odd, so that

$$\begin{aligned} n + \lambda t &= 2\left(n'' + \frac{t+1}{2} + \lambda'' t\right), \quad n'' = 1,\ 2,\ 3 \ldots\ldots \frac{t-3}{2} \\ &= 2\,(n''' + \lambda'' t), \quad n''' = \frac{t+1}{2},\ \ \frac{t+3}{2} \ldots\ldots t-1. \end{aligned}$$

And, moreover, by definition of v_n,

$$v_{2n} \equiv 2 v_n \pmod{t}.$$

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* Maschke, loc. cit., p. 176. † Ibid., p. 180.

If one makes $y_i^2 = x_i$, the system (56) will take the form

$$H_\kappa(x), \quad \psi_{n,\lambda,\mu}(x), \quad A^\nu(x).$$

But these invariants are identical in form with the system (37). The full system is, therefore, found in the reduced system

$$H_1(x), \quad \psi_n(x), \quad A(x). \tag{59}$$

THEOREM IX.—*The form system of the group* $\{T_1, T_2, S, \tau\}$ *is—*

1) *for* N_1 *even,* H_κ, $\psi_{n,\lambda,\mu}$, $A^{2\nu}$; *the full system is found by replacing* A *by* A_2 *in the full system of the group* $\{T_1, T_2, S\}$;

2) *for* N_1 *odd and* $\tau = \tau_4$ *the form system is given by* (55) *and the full system is contained in the reduced system* (58);

3) *for* N_1 *odd and* $\tau \neq \tau_4$ *the form system is*

$$H_\kappa(y^2), \quad \psi_{n,\lambda,\mu}(y^2), \quad A^\nu(y^2),$$

and the full system is found by replacing y *by* y^2 *in the full system of* $\{T_1, T_2, S\}$.

§10.—*The Invariant Forms of the Group* $\{T_1, T_2, s\}$.

If i, j, l be the subscripts of the k's in $T = (\omega_{N_1}^{k_i})$ and if the transposition (i, l) be denoted by $s_{i,l}$, the invariant forms of the group $\{T_1, T_2, s_{i,l}\}$ are rational integral functions of the forms

$$z_j^\lambda, \quad z_i^\mu + z_l^\mu, \quad (z_i^\alpha + z_l^\alpha) z_j^\beta, \quad z_i^{\alpha'} z_j^{\beta'} + z_i^{\beta'} z_j^{\alpha'} \text{ and } z_1 z_2 z_3.$$

The exponent λ satisfies the congruences (11). It has been found to be

$$\lambda \equiv 0 \left(\text{mod}\, \frac{N_1}{q_j}\right). \tag{60}$$

The exponent μ satisfies the four congruences

$$\left.\begin{aligned} k_i\mu \equiv k_l\mu \equiv 0 \ (\text{mod}\ N_1), \\ k_i'\mu \equiv k_l'\mu \equiv 0 \ (\text{mod}\ N_2), \end{aligned}\right\}$$
$$\therefore\ \mu \equiv 0 \ (\text{mod}\ N_1), \tag{61}$$

since $[k_i, k_l, N_1] = 1$.

In order that the form $(z_i^\alpha + z_l^\alpha) z_j^\beta$ shall be invariant, α and β must satisfy

the congruences

$$\left.\begin{aligned} k_i\alpha + k_j\beta &\equiv 0 \pmod{N_1}, \\ k_l\alpha + k_j\beta &\equiv 0 \pmod{N_1}, \\ k_i'\alpha + k_j'\beta &\equiv 0 \pmod{N_2}, \\ k_l'\alpha + k_j'\beta &\equiv 0 \pmod{N_2}. \end{aligned}\right\} \tag{62}$$

It follows immediately that

$$\alpha \equiv \beta \equiv 0 \pmod{N_2},$$

whence the congruences (62) reduce to

$$\left.\begin{aligned} k_i\alpha_1 + k_j\beta_1 &\equiv 0 \pmod{\bar{N}}, \\ k_l\alpha_1 + k_j\beta_1 &\equiv 0 \pmod{\bar{N}}, \end{aligned}\right\} \tag{63}$$

where $\qquad \alpha_1 = \alpha_1 N_2 \beta \equiv \beta_2 N_2.$

As before, put $\bar{N} = q_1 q_2 q_3 R = QR$, where $q_i = [k_i, \bar{N}]$.

The congruences (63) then reduce to

$$\left.\begin{aligned} q_i \varkappa\, \alpha_1' + \varkappa_j \beta_1' &\equiv 0 \pmod{R}, \\ q_l \varkappa_l \alpha_1' + \varkappa_j \beta_1' &\equiv 0 \pmod{R}, \end{aligned}\right\} \tag{64}$$

where $\qquad \alpha_1 = \alpha_1' Q, \quad \beta_1 = \beta_1' q_i q_l, \quad k_i = q_i \varkappa_i.$

If either q_i or q_l contains a prime factor ε which is found in R, the same factor must occur in β_1' and consequently in α_1'. When this factor is divided out, the resulting congruences will differ from (64) only in that the modulus will be $\frac{R}{l}$. If $\frac{R}{l}$ contains ε, this further factor is found in α and β also.

Let

$$Q_i = \frac{Q}{q_i} \qquad i = 1, 2, 3.$$

Also let P be the product of all the prime factors common to R and q_i and common to R and q_l, each one taken as often as it occurs in R, and let

$$\alpha_1 = \alpha_2 QP, \quad \beta_1 = \beta_2 Q_j P, \quad \bar{N} = QPR'. \tag{65}$$

The congruences (63) reduce to

$$\left.\begin{aligned} q_i \varkappa_i \alpha_2 + \varkappa_j \beta_2 &\equiv 0 \pmod{R'}, \\ q_l \varkappa_l \alpha_2 + \varkappa_j \beta_2 &\equiv 0 \pmod{R'}, \end{aligned}\right\} \tag{66}$$

in which the coefficients are prime to the modulus. To solve (66), let

$$t' = [k_i - k_l,\ R'],\quad k_i - k_l = s't' \text{ and } R' = r't'. \tag{67}$$

The congruences (66) will reduce to

$$\left.\begin{aligned} k_i\alpha_3 + \varkappa_j\beta_3 &\equiv 0 \pmod{t'}, \\ k_l\alpha_3 + \varkappa_j\beta_3 &\equiv 0 \pmod{t'}, \end{aligned}\right\} \tag{68}$$

where $$\alpha_2 = \alpha_3 r',\quad \beta_2 = \beta_3 r'. \tag{69}$$

The solution of (68) is

$$\left.\begin{aligned} \alpha_2 &\equiv n \quad (\mathrm{mod}\ t'), \\ \beta_3 &\equiv v'_n \quad (\mathrm{mod}\ t'), \\ v'\varkappa_j + k_i &\equiv 0 \quad (\mathrm{mod}\ t'), \\ v'_n &\equiv nv' \quad (\mathrm{mod}\ t'). \end{aligned}\right\} \tag{70}$$

The solution of (62) is, therefore,

$$\left.\begin{aligned} \alpha &= N_2\, Q\, Pr'\,(n + \lambda t'), \\ \beta &= N_2\, Q_j Pr'\,(v'_n + \mu t'), \end{aligned}\right\} \tag{71}$$
$$n = 0, 1, 2 \ldots\ t' - 1.$$

In order that the forms $z_i^{\alpha'} z_l^{\beta'} + z_l^{\alpha'} z_i^{\beta'}$ may be invariant, the following congruences must be true:

$$\left.\begin{aligned} k_i\alpha' + k_l\beta' \equiv k_l\alpha' + k_i\beta' &\equiv 0 \pmod{N_1}, \\ k_i\alpha' + k'_l\beta' \equiv k'_l\alpha' + k'_i\beta' &\equiv 0 \pmod{N_2}. \end{aligned}\right\} \tag{72}$$

These congruences reduce easily to

$$\left.\begin{aligned} k_i\alpha'_2 + k_l\beta'_2 &\equiv 0 \pmod{q_j R'}, \\ k_l\alpha'_2 + k_i\beta'_2 &\equiv 0 \pmod{q_j R'}, \end{aligned}\right\} \tag{73}$$

where $$\alpha' = N_2\, Q_j\, P\alpha'_2,\quad \beta' = N_2\, Q_j\, P\beta'_2. \tag{74}$$

We know that $$q_j t' = [k_i^2 - k_l^2,\quad q_j R'],$$

so that if $$\alpha'_2 = \alpha'_3 r',\quad \beta'_2 = \beta'_3 r', \tag{75}$$

we obtain $$\left.\begin{aligned} k_i\alpha'_3 + k_l\beta'_3 &\equiv 0 \pmod{q_j t'}, \\ k_l\alpha'_3 + k_i\beta'_3 &\equiv 0 \pmod{q_j t'}. \end{aligned}\right\} \tag{76}$$

By processes similar to those already employed, we find for the solution of (72),

$$\left.\begin{aligned}
\alpha' &= N_2\, Q_j\, Pr'\,(n + \lambda q_j t'),\\
\beta' &= N_2\, Q_j\, Pr'\,(v''_n + \mu q_j t'),\\
v'' k_l + k_i &\equiv 0 \pmod{q_j t'},\\
v''_n &\equiv n v'' \pmod{q_j t'},\\
n &= 0,\ 1,\ 2 \ldots\ldots q_j t' - 1.
\end{aligned}\right\} \tag{78}$$

Let $\vartheta' = N_2\, Q_j\, P\gamma'$, $q_j t' = t''$, $z_i^{\vartheta'} = x_i$. We have then

Theorem X.—*The invariant forms of the group* $\{T_1,\ T_2,\ S_{il}\}$ *are rational integral functions of*

$$\left.\begin{aligned}
& x_j^{\kappa t'}, \quad \sqrt[\vartheta']{(x_1 x_2 x_3)^{\nu}},\\
& \left(x_i^{q_j(n' + \lambda t')} + x_l^{q_j(n' + \lambda t')}\right) x_j^{v'_{n'}},\\
\text{and}\quad & x_i^{n'' + \lambda' t''}\, x_l^{v''_{n''} + \mu' t''} + x_i^{v''_{n''} + \mu' t''}\, x_l^{n'' + \lambda' t''},
\end{aligned}\right\} \tag{79}$$

where κ, λ, ν, λ', μ' *are positive integers.*

$v'_{n'}$ and $v''_{n''}$ *are defined by* (70) *and* (78) *and*

$$n' = 0,\ 1,\ 2 \ldots\ldots t',$$
$$n'' = 0,\ 1,\ 2 \ldots\ldots t''.$$

§11.—*The Quantities* v', v'' *and* t'.

The quantity v' is determined uniquely by either of the two congruences

$$\left.\begin{aligned}
v' k_j &\equiv - k_i \pmod{t'},\\
v' k_j &\equiv - k_l \pmod{t'}.
\end{aligned}\right\} \tag{80}$$

With the aid of the relation $\Sigma k \equiv 0 \pmod{t'}$ one finds

$$2v' \equiv q_j \pmod{t'}. \tag{81}$$

If t' is odd, v' is determined uniquely by (81). If t' is even, v' is either

$$v'_0 \text{ or } v'_0 + \frac{t'}{2}, \text{ where } v'_0 \equiv \frac{q_j}{2} \left(\operatorname{mod} \frac{t'}{2}\right). \tag{82}$$

The quantity v'' is determined uniquely by the congruence

$$v'' k_l \equiv - k_i \pmod{q_j t'}, \tag{83}$$

We know that $k_i^2 - k_i^2 \equiv 0 \pmod{q_j t'}$;

therefore, $$v''^2 - 1 \equiv 0 \pmod{q_j t'}. \tag{84}$$

The congruence (84) has at least one root for $q_j t' = 2$ and at least two roots for $q_j t' > 2$.*

From (84) we obtain easily

$$v'' + 1 \equiv 0 \pmod{t'} \tag{85}$$

and $$v'' - 1 \equiv 0 \pmod{q_j}. \tag{86}$$

From (85) and (86) it follows that $[q_j, t']$ is 1 or 2, while from (81), $(q_j, t') = 2$ when t' is even.

If, therefore, $[q_j, t] = 1$, v'' may be found from (86) and (87). It is

$$\left.\begin{aligned} v'' &= 1 + q_j \rho, \\ q_j\rho + 2 &\equiv 0 \pmod{t'}, \end{aligned}\right\} \tag{87}$$

where

If, however, $[q_j, t] = 2$, then v'' is one of the two numbers

$$1 + q_j\rho_0, \quad 1 + q_j\left(\rho_0 + \frac{t'}{2}\right),$$

where q_j is the smaller of the two roots of $q_j \rho + 2 \equiv 0 \pmod{t'}$, unless $t' = 2$.†
For $t' = 2$, (86) gives at once

$$v'' = 1.$$

These results may be stated as follows:

For t' odd, v' and v'' are given by (81) *and* (87). *For t' even and* >2, *v' is one of the numbers v'_0 or $v'_0 + \frac{t'}{2}$, where $v'_0 \equiv \frac{q_j}{2}\left(\text{mod}\, \frac{t'}{2}\right)$, and v'' is one* numbers $1 + q_j\rho_0$, $1 + q_j\left(\rho_0 + \frac{t'}{2}\right)$, *where ρ_0 is the smaller of the roots of* $q_j\rho + 2 \equiv 0 \pmod{t'}$. *For $t' = 2$, $v'' = 1$.*

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* Dirichlet, "Zahlentheorie," p. 88, ed. 1894.

† Serret, "Alg. Sup.", No. 292.

§12.—*The Full System of the Group* $\{T_1, T_2, S\}$.

For brevity let

$$\left.\begin{aligned} A' &= \sqrt[\vartheta']{(x_1 x_2 x_3)}. \\ \phi_{n,\lambda,\mu} &= x_i^{n+\lambda t''} x_l^{v'_n + \mu t''} + x_i^{v'_n + \mu t''} x_l^{n+\lambda t''}, \\ \chi_{n,\lambda} &= (x_i^{q_j(n+\lambda t')} + x_l^{q_j(n+\lambda t')}) x_j^{v'_n}. \end{aligned}\right\} \tag{88}$$

ϕ_n and χ_n will be written for $\phi_{n,0,0}$ and $\chi_{n,0}$, and where no ambiguity can arise, $\phi_{n,\lambda}$ and $\phi_{n,\mu}$ will be written for $\phi_{n,\lambda,0}$ and $\phi_{n,0,\mu}$.

The problem is then to find the full system for the set of forms

$$x_j^{\kappa t}, \quad A'^{\nu}, \quad \phi_{n,\lambda,\mu} \text{ and } \chi_{n,\lambda}.$$

$x_j^{t'}$ and A' evidently belong to the full system.

1). The form $x_i^{t'} x_l^{t'}$ is invariant, since $(x_i x_j x_l)^{t'}$ and $x_j^{t'}$ are both invariant. It is easily shown that $\phi_{t'} = 2x_i^{t'} x_l^{t'}$.

2). The forms $\chi_{n,\lambda}$ are expressible in terms of the forms $x_j^{t'}$, ϕ_n, $n = 1, 2, 3 \ldots\ldots t'' - 1$, χ_n, $n = 1, 2, 3 \ldots\ldots t' - 1$ and the form $\chi_{0,1} = x_i^{t''} + x_l^{t''}$. For we have

$$\chi_{0,1} \cdot \chi_{n,\nu} = \chi_{n,\nu+1} + \tfrac{1}{2} \phi_{t'}^{q_j} \cdot \chi_{n,\nu-1},$$

whence

$$\chi_{n,\nu+1} = \chi_{0,1} \cdot \chi_{n,\nu} - \tfrac{1}{2} \phi_{t'}^{q_j} \cdot \chi_{n,\nu-1}.$$

If, therefore, the proposition is true for $\lambda \leqq \nu$, it is true for $\lambda = \nu + 1$. But it is true for $\lambda = 1$ since, by multiplication,

$$\chi_{0,1} \cdot \chi_n = \chi_{n,1} + (x_i^{q_j t'} x_l^{q_j n} + x_l^{q t'} x_i^{q_j n}) x^{v'_n}.$$

If $q_j n > v'_n$, the last term contains the invariant factor $(x_1 x_2 x_3)^{v'_n}$ and the other factor is one of the set ϕ_n. If $q_j n < v'_n$, the factors of the last term are the invariant $(x_1 x_2 x_3)^{q_j n}$ and one of the set χ_n.

The case $q_j n = v_n$ cannot occur since, in such case,

$$\begin{aligned} q_j n - v'_n &\equiv n'(q_j - v') (\text{mod } t') \\ &\equiv nv' \; (\text{mod } t') \text{ by } (81) \\ &\equiv 0 \; (\text{mod } t'). \end{aligned}$$

But, by definition, $[v'.t''] = 1$; then $nv' \equiv 0 \pmod{t'}$ gives $n \equiv 0 \pmod{t'}$, which case is excluded.

The proposition is thus proven for all values of λ.

3). The forms $\phi_{n,\lambda,\mu}$ are expressible in terms of the forms

$$x_i^{t''} x_l^{t'}, \quad \chi_{0,1}, \quad \phi_{n,1,0} \text{ and } \phi_{v_n'',1,0}.$$

The proposition is evident for $\phi_{n,\lambda,\lambda}$. To fix the ideas, let $\lambda > \mu$, then, after the invariant factor $(x_i^{t''} x_l^{t''})^{\mu}$ is removed, it remains to consider the factor $\phi_{n,\lambda'}$, $\lambda' = \lambda - \mu$. One finds

$$\phi_{n,\nu} \cdot \chi_{0,1} = \phi_{n,\nu+1} + x_i^{t''} x_l^{t''} \cdot \phi_{n,\nu-1},$$

so that the assertion is true for $\lambda = \nu + 1$ if it is true for $\lambda \overline{\lessgtr} \nu$. It is true for $\lambda = 1$; hence true universally.

Similar considerations hold for the forms $\phi_{v_n'',\mu}$.

4). The forms $\phi_{n,1,0}$ and $\phi_{v_n'',1,0}$ are expressible in terms of $x_i^{t'} x_l^{t'}$, $\chi_{0,1}$ and the forms ϕ_n.

If one of these two forms is known, the other is known from the relation

$$\phi_n \cdot \chi_{0,1} = \phi_{n,1,0} + \phi_{v_n'',1,0}.$$

Let us consider the forms $\phi_{v_n'',1,0}$.

α). If $n = t'$, $v_n'' = t'$, and $\phi_{v_n'',1,0}$ breaks up into the two known forms $x_i^{t''} x_j^{t''}$ and $\chi_{0,1}$.

β). If $n > t'$, we may suppose that $\rho t' < n < (\rho + 1) t'$. Then

$$\phi_{v_n'',1,0} = (x_i^{t'} x_l^{t'})^{\rho} (x_i^{n-\rho t'} x_l^{v_n''+t''-nt'} + x_i^{v_n''+t''-\rho t} x_l^{n-\rho t'}). \tag{89}$$

When $v_n'' - \rho t' < 0$, the second factor on the right of (88) is one of the forms ϕ_n. If, however, $v_n'' - \rho t' > 0$, we have

$$\begin{aligned} v_n'' - \rho t' &\equiv (n - \rho t') v'' \,(\mathrm{mod}\ t'') \quad \text{by (86)} \\ &\equiv v_{n-\rho t'}'' \,(\mathrm{mod}\ t''). \end{aligned}$$

For the case under discussion

$$v_n'' - \rho t' = v_{n-\rho t'}''.$$

We have then

$$\phi_{v_n'',1,0} = (x_i^{t'} x_l^{t'})\, \phi_{v_n''-\rho t',1}.$$

The determination of the forms $\phi_{v_n'',1}$ is then made to depend upon the solution of the next case, viz.

γ). $n < t'$.

If $v''_n > t'$, we have at once

$$\phi_n \cdot \chi_{0,1} = (x_i^{t'} x_l^{t''})^\rho \phi_{n+t''-\rho t'} + \phi_{v''_n, 1, 0}.$$

If both n and v''_n are less than t', $\phi_n = \phi_{v''_n}$ is of degree t', since $n + v'_n \equiv n(v+1) (\text{mod } t'')$ is divisible by t' by (84). Moreover, n and v''_n are different for all values of n when t' is odd, and for all values except $n = \frac{t'}{2}$ when t' is even, since, if $n = v''_n$, we have

$$n(1-v) \equiv 0 \ (\text{mod } q_j t').$$

If t' is odd, $1 - v''$ contains q_j and no other factor of the modulus. If t' is event $1 - v''$ contains q_j and 2 by (85) and (86).

$\therefore$ $n = t'$ or $\frac{t'}{2}$, according as t' is odd or even. The corresponding form is $(x_i x_l)^{t'}$ or $(x_i x_l)^{\frac{t'}{2}}$. Consequently the form $\phi_{n,1}$ breaks up into the two known factors

$$(x_i x_l)^{t'} \text{ and } \chi_{0,1} \text{ or } (x_i x_l)^{\frac{t'}{2}} \text{ and } \chi_{0,1}.$$

If $n \neq v''_n$, $\phi_{n,1}$ is identical with some $\phi_{v''_n,1}$ when n and v'' are both less than t', so that we need consider only the cases where $n > v''_n$.

It may be shown that

$$\phi_n \cdot \phi_{n-v''_n} = \phi_{v''_n,1} + x_i^{2n-v''_n} x_l^{2v''_n - n + t''} + x_l^{2n-v''_n} x_i^{2v''_n - n + t''}. \tag{90}$$

If $2v''_n - n'' < 0$, the problem is solved, but if $2v''_n - n > 0$, (90) may be written

$$\phi_n \cdot \phi_{n-v''_n} = \phi_{v''_n,1} + \phi_{v''_{n_1},1}, \tag{91}$$

where it may be shown that $v''_{n_1} > v''_n$, and consequently $< n$ and

$$n_1 = 2v''_n - n \not> n.$$

The proof may be completed by induction.

5). The forms ϕ_n are expressible in terms of the first t' forms of the set.

For any $n > t'$, one may write $n = n_1 + \lambda t'$, $n_1 < t'$. Then

$$v''_n \equiv v''_{n_1} + \lambda t' \ (\text{mod } t'').$$

It follows directly that for $n > t'$,

$$\phi_n = (x_i x_l)^{\lambda t'} \cdot \phi_{n_1},$$

so that 5) is proven.

The lemmas 1), 2), 3), 4), 5) give the following theorem:

THEOREM XI.—*The invariant forms of the group* $\{T_1, T_2, s_{i,k}\}$ *are expressible rationally in terms of the* $2(t'+1)$ *forms*

$$\left.\begin{array}{l} x_j^{t'},\ \sqrt[\vartheta']{x_1 x_2 x_3}, \\ \chi_n = (x_i^{q_j n} + x_l^{q_j n})\, x_j^{v'_n}, \qquad \phi_n = x_i^n x_l^{v''_n} + x_l^n x_i^{v''_n}, \end{array}\right\} \tag{92}$$
$$n = 1, 2, 3 - t',$$

where t', v'_n, v''_n, ϑ' *and* q_j *have the meanings assigned in* §10, *and* $x_i = z_i^{\vartheta'}$. *The full system will consist of the forms* $x_j^{t'}$, $x_i^{t''} + x_l^{t''}$, *the forms* χ_n, *for which* $n_1 + n_2 = n$ *and* $v'_{n_1} + x'_{n_2} = n$, *are not both true and the forms* ϕ_n, *for which* $n \overline{\gtrless} t'$, *and* $n_1 + n_2 = n$ *and* $v''_{n_1} + v''_{n_2} = v''_n$ *are not both true.*

§13.—*The Invariant Forms of the Group* $\{T_1, T_2, s_{ik}, \tau\}$.

The invariant forms of the group $\{T_1, T_2, S_{ik}, \tau\}$ are all found among the forms of the group $\{T_1, T_2, s_{ik}\}$. It is clear moreover that τ either leaves any given invariant of the latter group unchanged or simply changes its sign. The invariant forms of the group $\{T_1, T_2, s_{ik}, \tau\}$ will then be found by imposing proper conditions upon the exponents $\varkappa$, λ, μ, λ', μ', ν of the forms (79) and adding to the forms thus obtained certain products of forms which change sign with respect to τ.

There are several cases with subcases depending upon the character of ϑ', t', q_j and t''. The results are here given without proof.

Case I. ϑ' even.

The form system is the system obtained from (92) by excluding odd powers of $\sqrt[\vartheta']{x_1 x_2 x_3}$, and in the full system $\sqrt[\vartheta']{x_1 x_2 x_3}$ is replaced by $\sqrt[\vartheta']{(x_1 x_2 x_3)^2}$.

Case II. ϑ' odd, $t'' = q_j t'$ even.

There are several subcases depending on the character of t' and q_j and the particular τ* that enters into the group.

* See §7 (49), above.

1). t' even.

1a). $\tau = \tau_j$ or τ_4.

In the forms $\chi_{n,\lambda}$, n must be even and the remaining condition to be imposed upon the exponents of (79) is $\nu \equiv 0 \pmod 2$. Besides the forms thus obtained, one has also to include the forms

$$\sqrt[\vartheta']{x_1 x_2 x_3} \cdot \chi_{n,\lambda}, \quad \chi_{n'\lambda}, \cdot \chi_{n'',\lambda}, \quad n,\ n' \text{ and } n'' \text{ odd.}$$

There is a reduced system consisting of the forms

$$x_j^{t'}, \quad \sqrt[\vartheta']{x_1 x_2 x_3}, \quad \phi_n, \quad (n = 1,\ 2,\ 3\ \ldots\ldots t'), \quad \chi_{2n}, \quad n = 1,\ 2\ \ldots\ldots \frac{t'}{2},$$

$$\sqrt[\vartheta']{x_1 x_2 x_3} \cdot \chi_{2n-1} \quad n = 1,\ 2\ \ldots\ldots \frac{t'}{2}, \quad \chi_{n_1} \cdot \chi_{n_2}, \quad n_1 \text{ and } n_2 \text{ both odd.}$$

1b). $\tau = \tau_i$ or τ_l.

n must be even in the forms $\phi_{n,\lambda,\mu}$ and we must have also $\nu \equiv 0 \pmod 2$.

The forms $\sqrt[\vartheta']{x_1 x_2 x_3} \cdot \phi_{n,\lambda,\mu}$, n odd and $\phi_{n'\lambda,\mu} \cdot \phi_{n'',\lambda,\mu}$, n' and n'' both odd, are to be included.

For a reduced system, we have

$$x_j^{t'}, \quad \phi_{2n}, \quad n = 1,\ 2,\ 3\ \ldots\ldots \frac{t'}{2}, \quad \chi_n, \quad n = 1,\ 2\ \ldots\ldots t',$$

$$\sqrt[\vartheta']{x_1 x_2 x_3} \cdot \phi_{2n-1}, \quad n = 1,\ 2\ \ldots\ldots \frac{t'}{2}, \quad \phi_{n'} \cdot \phi_{n''}, \quad n' \text{ and } n'' \text{ both odd.}$$

2). t' odd, q_j even.

2a). $\tau = \tau_j$ or τ_4.

The conditions to be imposed upon the exponents are $\varkappa \equiv \rho \equiv v'_n \equiv 0 \pmod 2$. The forms

$$x_j^{t'} \cdot \chi_{n,\lambda}, \quad \sqrt[\vartheta']{x_1 x_2 x_3} \cdot \chi_{n,\lambda}, \quad v'_n \text{ odd}, \quad \chi_{n',\lambda} \cdot \chi_{n'',\lambda},$$

n' and n'' both odd, are to be included.

There is a reduced system consisting of the forms

$$x_j^{2t'}, \quad \sqrt[\vartheta']{(x_1 x_2 x_3)^2}, \quad \phi_n, \quad n = 1,\ 2,\ 3\ \ldots\ldots t', \quad \chi_n, \quad v'_n \text{ even},$$

$$x_j^{t'} \cdot \chi_n \text{ and } \sqrt[\vartheta']{x_1 x_2 x_3} \cdot \chi_n, \quad v'_n \text{ odd},$$

$$\chi_{n'} \cdot \chi_{n''}, \quad n' \text{ and } n'' \text{ both odd.}$$

2b). $\tau = \tau_i$ or τ_l.

The form-system is identical with that in the case 1b) above.

Case III. ϑ' odd and t'' odd.

1). $\tau = \tau_j$.

The form-system is identical with that of the case 1a) under II.

2). $\tau = \tau_4$.

The conditions are

$$\varkappa \equiv \nu \equiv n + v'_n + \lambda \equiv n + v''_n + \lambda + \mu \equiv 0 \pmod 2.$$

To the forms thus obtained must be added the forms

$$x_j^{t'} \cdot \sqrt[\vartheta']{x_1 x_2 x_3}, \quad x_j^{t'} \cdot \phi_{n,\lambda,\mu}, \quad x_j^{t'} \cdot \chi_{n,\lambda}, \quad \sqrt[\vartheta']{x_1 x_2 x_3} \cdot \chi_{n,\lambda}, \quad \sqrt[\vartheta']{x_1 x_2 x_3} \cdot \phi_{n,\lambda,\mu},$$
$$\phi_{n,\lambda,\mu} \cdot \chi_{n,\lambda}, \quad \phi_{n',\lambda,\mu} \cdot \phi_{n'',\lambda,\mu}, \quad \chi_{n',\lambda} \cdot \chi_{n'',\lambda},$$

for which the conditions $n + v'_n + \lambda \equiv n + v''_n + \lambda + \mu \equiv 1 \pmod 2$ hold. There exists a reduced system consisting of the forms

$$x_j^{2t'}, \quad \sqrt[\vartheta']{(x_1 x_2 x_3)^2}, \quad \chi_n, \quad n + v'_n \text{ even}, \quad \phi_n, \quad n + v''_n \text{ even},$$

together with the product made up by taking two distinct factors from the forms

$$x_j^{t'}, \quad \sqrt[\vartheta']{x_1 x_2 x_3}, \quad \chi_n, \; n + v'_n \text{ odd and } \phi_n, \; n + v''_n \text{ odd}.$$

3). $\tau = \tau_i$ or τ_l.

The conditions are

$$\varkappa \equiv \nu \equiv n' + \lambda \equiv n'' + \lambda \equiv v''_{n''} + \mu \equiv 0 \pmod 2,$$

where n' and n'' belong to $\chi_{n,\lambda}$ and $\phi_{n,\lambda,\mu}$ respectively.

To these forms must be added the forms $x_j^{t'} \cdot \sqrt[\vartheta']{x_1 x_2 x_3}$, together with $x_j^{t'} \cdot \chi_{\mu,\lambda}$, $x_j^{t'} \cdot \phi_{n,\lambda,\mu}$, $\sqrt[\vartheta']{x_1 x_2 x_3} \cdot \chi_{n,\lambda}$, $\sqrt[\vartheta']{x_1 x_2 x_3} \cdot \phi_{n,\lambda,\mu}$, $\chi_{n',\lambda} \cdot \chi_{n'',\lambda}$, $\phi_{n',\lambda,\mu} \cdot \phi_{n'',\lambda,\mu}$ and $\chi_{n,\lambda} \cdot \phi_{n,\lambda,\mu}$, for which

$$n' + \lambda \equiv n'' + \lambda \equiv v''_{n''} + \mu \equiv 1 \pmod 2.$$

There is a reduced system consisting of the following forms:

$$x_j^{2t'}, \; \sqrt[\vartheta']{(x_1 x_2 x_3)^2}, \; \chi_n \; n \text{ even}, \; \phi_n, \; n \text{ and } v''_n \text{ even},$$

together with the products taken two at a time of the forms x_j^{t}, $\sqrt[\vartheta']{x_1 x_2 x_3}$, χ_n n odd, ϕ_n, n and v''_n not both even.

§14.—*The Invariant Forms of the Group* $\{T_1, T_2, S, s\}$.

If S and s be the generators of the symmetric group of three elements, the invariant forms of the group $\{T_1, T_2, S, s\}$ are rational integral functions of the symmetric functions

$$\Sigma z^{\kappa}, \quad \Sigma z_1^{\alpha} z_2^{\beta}, \quad (z_1 z_2 z_3)^{\nu}.$$

The exponent ν is any integer, while $\kappa \equiv 0 \pmod{N_1}$.

The conditions that the form $\Sigma z_1^{\alpha} z_2^{\beta}$ shall be invariant are given by six congruences of the form

$$k_i \alpha + k_j \beta \equiv 0 \pmod{N_1} \tag{93}$$

and six of the form

$$k_i' \alpha + k_j' \beta \equiv 0 \pmod{N_2}, \tag{94}$$

in which the numbers i and j are any arrangement of two of the numbers 1, 2, 3.

From the two congruences

$$k_i \alpha + k_j \beta \equiv 0 \pmod{N_2}$$

and

$$k_i' \alpha + k_j' \beta \equiv 0 \pmod{N_2},$$

one finds

$$\alpha \equiv 0 \pmod{N_2}, \quad \beta \equiv 0 \pmod{N_2}.$$

Let

$$\alpha = N_2 \alpha_1, \quad \beta = N_2 \beta_1, \tag{95}$$

then the twelve congruences (93) and (94) reduce to six of the form

$$k_i \alpha_1 + k_j \beta_1 \equiv 0 \pmod{\bar{N}}. \tag{96}$$

By reason of the relation $\Sigma k \equiv 0 \pmod{N_1}$, the six congruences (96) reduce to four, which may be written as follows:

$$\left.\begin{aligned} k_1 \alpha_1 + k_2 \beta_1 &\equiv 0 \pmod{\bar{N}}, \\ k_2 \alpha_1 - (k_1 + k_2) \beta_1 &\equiv 0 \pmod{\bar{N}}, \end{aligned}\right\} \tag{97}$$

$$\left.\begin{aligned} k_2 \alpha_1 + k_1 \beta_1 &\equiv 0 \pmod{\bar{N}}, \\ (k_1 + k_2) \alpha_1 + k_2 \beta_1 &\equiv 0 \pmod{\bar{N}}\cdot \end{aligned}\right\} \tag{98}$$

in which the notation is identical with that in the congruences (27).

Comparing (97) and (98) with (27), one has at once the solutions sought, viz. For (97),

$$\left.\begin{aligned}
&\left.\begin{aligned}\alpha &= N_2\,Qr(n+\lambda t),\\ \beta &= N_2\,Qr(v_n+\mu t)\end{aligned}\right\}\text{(a)}\\
\text{or}\quad&\left.\begin{aligned}\alpha &= N_2\,Qr(w_n+\lambda' t),\\ \beta &= N_2\,Qr(n+\mu' t),\end{aligned}\right\}\text{(b)}\\
\text{and for (98),}\quad&\\
&\left.\begin{aligned}\alpha &= N_2\,Qr(v_n+\lambda t),\\ \beta &= N_2\,Qr(n+\mu t),\end{aligned}\right\}\text{(c)}\\
\text{or}\quad&\left.\begin{aligned}\alpha &= N_2\,Qr(n+\lambda' t),\\ \beta &= N_2\,Qr(w_n+\mu' t).\end{aligned}\right\}\text{(d)}
\end{aligned}\right\}\tag{99}$$

If one compares (a) and (d) or (b) and (c) of (99) the following condition is obtained for n, viz.

$$n(v-w)\equiv 0 \pmod{t'}. \tag{100}$$

Two cases arise:

Case I. $[(v-w),\,t]=1$.

The congruence (100) has no solution except $n\equiv 0 \pmod{t}$, and, consequently, (93) and (94) have no solution except

$$\alpha\equiv\beta\equiv 0 \pmod{N_1},$$

The invariant forms are then symmetric functions of $z_1^{N_1}$, $z_2^{N_1}$, $z_3^{N_1}$, together with powers of $z_1 z_2 z_3$. The full system is

$$\Sigma z_1^{N_1},\quad \Sigma z_1^{N_1} z_2^{N_1},\quad z_1 z_2 z_3. \tag{101}$$

Case II. $[v-w,\,t]\neq 1$.

It was shown in §§5, 7, that if $[(v-w),\,t]\neq 1$, then $[(v-w),\,t]=3$. If $v-w=3m$ and $t=3s$ one finds

$$n\equiv 0 \pmod{s}.$$

Let

$$n=n_1 s;$$

then since

$$v_n=v_{n_1 s}\equiv n_1 s v \pmod{3s},$$

$v_{n_1 s}$ is divisible by s.

Let

$$\frac{v_{n_1 s}}{s}=\bar{v}_{n_1},$$

We find $\bar{v}_{n_1} \equiv n_1 v \pmod 3$.

It follows immediately that the solutions which satisfy the twelve congruences (93) and (94), are of the form

$$\left.\begin{aligned} \alpha &= \vartheta''(n + 3\lambda), \\ \beta &= \vartheta''(\bar{v}_n + 3\mu), \end{aligned}\right\} n = 0,\ 1,\ 2, \tag{102}$$

when $\vartheta'' = N_2\, Qrs = \vartheta s = \dfrac{N_1}{3}$.

Let $z_i^{\vartheta''} = \xi_i$. The invariant forms sought are then

$$\Sigma\xi_1^{3\kappa}, \quad \Sigma\xi_1^{n+3\lambda}\Sigma\xi_2^{\bar{v}n+3\mu}, \quad (\xi_1\xi_2\xi_2)^{\frac{3\nu}{N_1}} \tag{103}$$

where $n = 0, 1, 2$, and $\kappa, \lambda, \mu, \nu$ are arbitrary integers.

Furthermore, the congruence

$$x^2 - x + 1 \equiv 0 \pmod t,$$

of which v is a root, may be written in the present case

$$(2x - 1)^2 + 3 \equiv 0 \pmod{3s}.$$

It follows that $2v - 1 \equiv 0 \pmod 3$

and that $v \equiv 2 \pmod 3$.

Evidently $v_1 = 2$ and $v_2 = 1$.

The set of forms $\Sigma\xi_1^{1+3\lambda}\xi_2^{2+3\mu}$ is identical with the set $\Sigma\xi_1^{2+3\lambda}\xi_2^{1+3\mu}$.
We may then write the forms (103) as follows:

$$\Sigma\xi_1^{3\kappa}, \quad \Sigma\xi_1^{n+3\lambda}\xi_2^{2n+3\mu}, \quad (\xi_1\xi_2\xi_3)^{\frac{3\nu}{N_1}}, \qquad n = 0,\ 1. \tag{104}$$

The results may be summed up in the following:

Theorem XII.—*If* $t \not\equiv 0 \pmod 3$, *the invariant forms of the group* $\{T_1, T_2, S, s\}$ *are given by*

$$\Sigma\xi_1^{3\kappa}, \quad \Sigma\xi_1^{3\lambda}\xi_2^{3\mu}, \quad \sqrt[N_1]{(\xi_1\xi_2\xi_3)^{3\nu}}.$$

If $t \equiv 0 \pmod 3$, *they are given by*

$$\Sigma\xi_1^{3\kappa}, \quad \Sigma\xi_1^{n+3\lambda}\xi_2^{2n+3\mu}, \quad (n = 0, 1), \quad \sqrt[N_1]{(\xi_1\xi_2\xi_3)^{3\nu}}.$$

It remains to find the full system of the system (104). Let

$$\Sigma\xi_1^3 = C_1, \quad \Sigma\xi_1^3\xi_5^3 = C_2, \quad (\xi_1\xi_2\xi_3)^3 = C_3, \quad \Sigma\xi_1\xi_2^2 = D. \tag{105}$$

We have only to examine the forms $\Sigma\xi_1^{1+3\lambda}\xi_2^{2+3\mu}$, since all other forms of the system are expressible in terms of C_1, C_2, $C_3^{\frac{1}{N_1}}$. Let $\Sigma\xi_1^{1+3\lambda}\xi_2^{2+3\mu} = D_{\lambda,\mu}$. The forms $D_{\lambda,\mu}$ are expressible rationally in terms of C_1, C_2, C_3 and D. For, suppose the statement be true for all forms of order $3n$ in the ξ's or less; then the $n+1$ relations

$$\begin{aligned} \Sigma\xi_1^6 . D &= D_{3,n-1} + D_{0,n} \quad + C_3^{\frac{1}{3}} D_{n-2,1}, \\ C_1 . D_{\lambda,\mu} &= D_{\lambda+1,\mu} + D_{\lambda,\mu+1} + C_3 D_{\lambda-1,\mu-1}, \end{aligned}$$

for which $\lambda + \mu = n - 1$ suffice to determine the $n+1$ forms of order $3(n+1)$ in the ξ's. But the statement is easily seen to be true for $n = 1$ and for $n+3$. It is therefore true generally.

THEOREM XIII.—*If* $t \not\equiv 0$ (*mod* 3), *the full system of the group* $\{T_1, T_2, S, s\}$ *is* C_1, C_2, $C_3^{\frac{1}{N_1}}$; *if* $t \equiv 0$ (*mod* 3), *it is* C_1, C_2, $C_3^{\frac{1}{N_1}}$, D.

Between the four forms of the full system C_1, C_2, $C_3^{\frac{1}{N_1}}$, D, there exists the single relation

$$D^3 = 3C_2 . D + 9C_3 + C_1C_2 + 3C_3^{\frac{1}{3}}(2C_2 + C_1D) + 3C_3^{\frac{2}{3}}(D + 2C_1). \quad (106)$$

§15.—*The Invariant Forms of the Group* $\{T_1, T_2, S, s, \tau\}$.

To find the full system of the group $\{T_1, T_2, S, s, \tau\}$, one has only to impose proper conditions upon the exponents occurring in the system of Theorem XII, and to add such products, two at a time, of forms belonging to the group $\{T_1, T_2, S, s\}$ as undergo no change except a change of sign when operated upon by τ.

The systems of invariants of Theorem XII, written out in full, are

for $t \not\equiv 0$ (mod 3),

$$\Sigma z_1^{\kappa N_1}, \quad \Sigma z_1^{\lambda N_1} z_2^{\mu N_1}, \quad (z_1 z_2 z_3)^{\nu}; \quad (107)$$

for $t \equiv 0$ (mod 3),

$$\Sigma z_1^{\kappa N_1}, \quad \Sigma z_1^{\vartheta''(n+3\lambda)} z_2^{\vartheta''(2n+3\mu)}, \quad n = 0, 1, \quad (z_1 z_2 z_3)^{\nu}. \quad (108)$$

The conditions to be imposed, as is easily seen, are given by the following tables:

I. $t \not\equiv 0 \pmod 3$:

$$1).\ N_1 \text{ even}\begin{cases}\alpha) & \tau = \tau_4, \quad \nu \equiv 0 \pmod 2,\\ \beta) & \tau \not\equiv \tau_4, \quad \nu = 0 \pmod 2.\end{cases}$$

$$2).\ N_1 \text{ odd}\begin{cases}\alpha) & \tau = \tau_4, \quad \kappa \equiv \lambda + \mu \equiv \nu \equiv 0 \pmod 2,\\ \beta) & \tau \not\equiv \tau_4, \quad \kappa \equiv \lambda \equiv \mu \equiv \nu \equiv 0 \pmod 2.\end{cases}$$

II. $t \equiv 0 \pmod 3$:

$$1).\ N_1 \text{ even}\begin{cases}\alpha) & \tau = \tau_4, \quad \nu \equiv 0 \pmod 2,\\ \beta) & \tau \not\equiv \tau_4, \quad \nu \equiv 0 \pmod 2.\end{cases}$$

$$2).\ N_1 \text{ odd}\begin{cases}\alpha) & \tau = \tau_4, \quad \kappa \equiv n + \lambda + \mu \equiv \nu \pmod 2,\\ \beta) & \tau \not\equiv \tau_4, \quad \kappa \equiv n + \lambda \equiv \mu \equiv \nu \pmod 2.\end{cases}$$

If, as before, the substitution $z_i^{\vartheta''} = \xi_i$ be made, the results obtained may be given by the following:

THEOREM XIV.—*The invariant forms of the group* $\{T_1, T_2, S, s, \tau\}$ *are—*

for $t \not\equiv 0$ (*mod* 3),

$$\Sigma\xi_1^{3\kappa}, \quad \Sigma\xi_1^{3\lambda}\xi_2^{3\mu}, \quad (\xi_1\xi_2\xi_3)^\nu,$$

the exponents subject to the conditions of table I;

for $t \equiv 0$ (*mod* 3),

$$\Sigma\xi_1^{3\kappa}, \quad \Sigma\xi_1^{\vartheta''(n+3\lambda)}\xi_2^{\vartheta''(2n+3\lambda)}, \quad (n = 0,\ 1), \quad (\xi_1\xi_2\xi_3)^\nu,$$

the exponents subject to the conditions of Table II, and in both cases when $\tau = \tau_4$, *the products composed of an even number of factors of odd order invariant with respect to the group* $\{T_1, T_2, S, s\}$ *must be added.*

§16.—*The Full Systems for the Group* $\{T_1, T_2, S, s, \tau\}$.

In order to abbreviate the work of finding the full systems for the cases given above, the following notation, part of which has already been used, will be adopted. We put

$$\left.\begin{array}{lll} C_1 = \Sigma\xi_1^3, & C_2 = \Sigma\xi_1^3\xi_2^3, & C_3 = (\xi_1\xi_2\xi_3)^3,\\ D = \Sigma\xi_1\xi_2^2, & E = \Sigma\xi_1^4\xi_2^2, & F = \Sigma\xi_1\xi_2^5,\\ G = \Sigma\xi_1^6\xi_2^6, & K = (\xi_1\xi_2\xi_3)^{\frac{3}{N_1}}, & \Sigma\xi_1^3 = C_3^{\frac{1}{N_1}}.C_1,\\ L = C_1^{\frac{1}{N_1}}.D, & S_2 = \Sigma\xi_1^6, & M = C_3^{\frac{1}{N_1}}.D. \end{array}\right\} \qquad (109)$$

Cases 1) and 2). $t \not\equiv 0$ (mod 3), N even, τ any τ.

The full system for both cases is clearly

$$C_1, \quad C_2, \quad C_3^{\frac{2}{N_1}}. \tag{110}$$

Case 3). N_1 odd, $t \not\equiv 0 \pmod 3$, $\tau = \tau_4$.

The forms belonging to the group $\{T_1, T_2, S, s, \tau\}$ are also invariant with respect to the group $\{T_1, T_2, S, s\}$. Every invariant of the latter group is expressible rationally in terms of C_1, C_2, $C_3^{\frac{1}{N_1}}$. This fact may be expressed by the equation

$$\phi(z_1, z_2, z_3) = \Sigma m_{\alpha, \beta, \gamma} C_1^{\alpha} C_2^{\beta} C_3^{\frac{\gamma}{N_1}}.$$

The form C_2 is an invariant of the group $\{T_1, T_2, S, s, \tau\}$. The even powers of C_1 and $C_3^{\frac{1}{N}}$ are expressible in terms of C_2, $C_3^{\frac{2}{N_1}}$ and $S_2 = \Sigma \xi_1^6$. It follows immediately that ϕ is expressible rationally in terms of

$$C_2, \quad C_3^{\frac{2}{N_1}}, \quad S_2 \text{ and } K, \tag{111}$$

and it is apparent that these forms (111) constitute the full system. Between the forms of the full system there exists the single relation

$$K^2 = C_3^{\frac{2}{N_1}} (S_2 + 2C_2). \tag{112}$$

4). $t \not\equiv 0 \pmod 3$, N_1 odd, $\tau \neq \tau_4$.

Since the conditions are $\varkappa \equiv \lambda \equiv \mu \equiv \nu \equiv 0 \pmod 3$, the system of forms is given by

$$\Sigma (\xi_1^2)^{3\varkappa}, \quad \Sigma (\xi_1^2)^{3\lambda} (\xi_2^2)^{3\mu}, \quad \sqrt[\vartheta'']{(\xi_1^2 \xi_2^2 \xi_3^2)^{\nu}}.$$

It is at once evident that the full system is

$$S_2, \quad G, \quad C_3^{\frac{2}{N_1}}. \tag{113}$$

5) and 6). $t \equiv 0 \pmod 3$, N_1 even, $\tau = \tau_4$ or $\tau \neq \tau_4$.

The system of forms differs from the system of the group $\{T_1 T_2 S, s\}$ only in the exclusion of odd powers of $C_3^{\frac{1}{N_1}}$. The forms $\Sigma \xi_1^{3\varkappa}$, $\Sigma \xi_1^{n+3\lambda} \xi_2^{2n+3\mu}$ are expressible in terms of C_1, C_2, C_3, and C_3 is, in the present case, an even power of $C_3^{\frac{1}{N_2}}$. Therefore, the full system is

$$C_1, \quad C_2, \quad D, \quad C_3^{\frac{2}{N_1}}. \tag{114}$$

The relation existing between the four forms (114) is the relation (106) which may be written in the form

$$D^3 = 3C_2D + 9\left(C_3^{\frac{2}{N_1}}\right)^{\frac{N_1}{2}} + C_1C_2 + 3\left(C_3^{\frac{2}{N_1}}\right)^{\frac{N_1}{6}}(2C_2 + C_1D) + 3\left(C_3^{\frac{2}{N_1}}\right)^{\frac{N_1}{3}}(D + 2C_1). \qquad (115)$$

7). $t \equiv 0 \pmod 3$, N_1 odd, $\tau = \tau_4$.

By a process similar to that used in 3), it is found that the forms of the system may be expressed rationally in terms of the seven forms

$$C_1^2,\quad C_1D,\quad C_1C_3^{\frac{1}{N_1}},\quad C_2,\quad D^2,\quad C_3^{\frac{1}{N_1}}D \text{ and } C_3^{\frac{2}{N_1}}.$$

Between these forms and the forms E, F, K and S_2, there exist the following relations:

$$\left.\begin{aligned} C_1^2 &= S_2 + 2C_2, \\ C_1D &= E + F + C_3^{\frac{1}{3}-\frac{1}{N_1}}.L, \\ D^2 &= E + 2C_3 + 2C_2^{\frac{1}{3}-\frac{1}{N_1}}(K + L) + 6C_3^{\frac{2}{3}}, \end{aligned}\right\} \qquad (116)$$

By means of the relations (116), the forms C_1^2, CD, D^2 may be replaced by S_2, F, G respectively. No other relations of the sixth order in the ξ's exist. We may then choose for the full system the forms

$$S_2,\quad C_2,\quad C_3^{\frac{2}{N_1}},\quad E,\quad F,\quad K,\quad L. \qquad (117)$$

The following relations hold for the forms (117):

$$\left.\begin{aligned} K^2 &= C_3^{\frac{2}{N_1}}(S_2 + 2C_2), \\ L^2 &= C_3^{\frac{2}{N_1}}\left[E + 2C_2 + 2C_3^{\frac{N_1}{2}\cdot\frac{N_1-3}{2}}(K + L) + 6C_3^{\frac{2}{3}}\right], \\ E^3 &= S_2(C_2^3 + 3C_3^{\frac{2}{3}}.E + 6C_3^{\frac{4}{3}}) \\ &\quad - 2C_3^{\frac{2}{N_1}\left(\frac{N_1-1}{2}\right)}K(3E + S_2 + 3C_3^{\frac{2}{3}}) \\ &\quad + 3C_3^{\frac{2}{3}}(3C_3^{\frac{2}{3}} + C_3^{\frac{2}{3}}E + 2C_2^2) + 3C_2^2E, \\ F^2 &= \left[E + 2C_2 + 2C_3^{\frac{2}{N_1}\cdot\frac{N_1-3}{6}}(K + L) + 6C_3^{\frac{2}{3}}\right] \\ &\quad \times\left[S_2 + 2C_2 + C_3^{\frac{2}{3}} - 2C_3^{\frac{2}{N_1}\cdot\frac{N_1-3}{6}}K\right] - E(E + 2F). \end{aligned}\right\} \qquad (118)$$

8). $t \equiv 0 \pmod 3$, N_1 odd, $\tau \neq \tau_4$.

For $n \gtreqless 2$ the $n+1$ equations

$$\Sigma\xi_1^6, \quad \Sigma\xi_1^{1+3(2\lambda+1)}\xi_2^{6\mu} = \Sigma\xi_1^{1+3(2\lambda+3)}\xi_2^{2+6\mu} + \Sigma\xi_1^{1+3(2\lambda+1)}\xi_2^{2+6(\mu+1)} + C_3^{\frac{1}{3}}\Sigma\xi_1^4\xi_2^{2+6\lambda}\xi_3^{6\mu},$$

$$\lambda = 0, 1, 2, 3 \ldots . n-1, \quad \lambda + \mu = n-1,$$

$$\Sigma\xi_1^6\xi_2^6 . \Sigma\xi_1^4\xi_2^{6(n-2)} = \Sigma\xi_1^4\xi_2^{2+6(n-1)} + C_3^{\frac{1}{3}}\Sigma\xi_1^{2+6(n-2)} + C_3^{\frac{1}{3}} . \Sigma\xi_1^4\xi_2^8\xi_2^{6(n-2)}$$

suffice to determine the $n+1$ forms

$$\Sigma\xi_1^{1+3(2\lambda+1)}\Sigma\xi_2^{2+6\mu}, \qquad \lambda = 0, 1, 2 \ldots . n, \quad \lambda + \mu = n,$$

of order $6(n+1)$ in the ξ's in terms of the forms of order $6n$ or lower. It is easily shown that the forms of orders 8 and 12 in the ξ's are expressible in terms of the forms

$$S_2, \quad G, \quad E, \quad C_3^{\frac{2}{N_1}}. \tag{119}$$

It follows immediately that these four forms constitute the full system.

The four forms of the full system are bound by the relation

$$E^3 = 3EG + 9C_3^2 + S_2G + 3C_3^{\frac{2}{3}}(2G + S_2E) + 3C_3^{\frac{4}{3}}(E + 2S_2). \tag{120}$$

The results just obtained give the following:

THEOREM XV.—*The full system of the groups* $\{T_1, T_2, S, s, \tau\}$ *are given as follows:*

For $t \not\equiv 0 \pmod 3$:

1) N_1 *even*, $\tau = \tau_4$, C_1, C_2, $C_3^{\frac{2}{N_1}}$.

2) " $\tau \neq \tau_4$, C_1, C_2, $C_3^{\frac{2}{N_1}}$.

3) N_1 *odd*, $\tau = \tau_4$, S_2, C_2, $C_3^{\frac{2}{N_1}}$, K.

4) " $\tau \neq \tau_4$, S_2, G, $C_3^{\frac{2}{N_1}}$.

For $t \equiv 0 \pmod 3$:

5) N_1 *even*, $\tau = \tau_4$, C_1, C_2, D, $C_3^{\frac{2}{N_1}}$.

6) " $\tau \neq \tau_4$, C_1, C_2, D, $C_3^{\frac{2}{N_1}}$.

7) N_1 *odd*, $\tau = \tau_4$, S_2, C_2, $C_3^{\frac{2}{N_1}}$, E, F, K, L.

8) " $\tau \neq \tau_4$, S_2, G, E, $C_3^{\frac{2}{N_1}}$.

where the forms C_1, C_2, C_3, E, F, G, K, L, S_2 *are defined by* (109).

The relations existing in those cases where more than three forms belong to the full system are, for case 3), (112) ; *for cases* 5) *and* 6), (115) ; *for case* 7, (118) ; *for case* 8), (120)

CHAPTER III.

The Orders of the Principal Ternary Monomial Groups.

§17.—*The Order of the Group* $\{T_1, T_2, S\}$.

Let $$U_i = ST_iS^{-1} = (\omega_{N_i}^{k_3}, \omega_{N_i}^{k_3}, \omega_{N^i}^{k_{1i}}), \qquad i = 1, 2.$$

The substitution U_2 belongs to the group $\{T_1, T_2\}$, for, from the condition

$$T_1^{\alpha} T_2^{\beta} = U^{\delta},$$

one has for the determination of α, β, δ the two independent congruences

$$\left.\begin{array}{l} \bar{N}\, k'_2\delta \equiv k_1\alpha + \bar{N}\, k'_1\beta, \\ \bar{N}\, k_2\delta \equiv k_2\alpha + \bar{N}\, k'_2\beta, \end{array}\right\} \pmod{N_1}. \tag{121}$$

The congruences (121) reduce at once to

$$\left.\begin{array}{l} k_1\alpha_1 + k'_1\beta \equiv k'_2\delta, \\ k_2\alpha_1 + k'_2\beta \equiv k'_3\delta, \end{array}\right\} \pmod{N_2}, \tag{122}$$

where $\alpha = \alpha_1\bar{N}$.

By hypothesis $[(k_1\, k'_2),\, N_2] = 1$, so that one may find α_1 and β from (122) whatever value may be assigned to δ. We have then

$$U_2 = T_1^{\alpha} T_2^{\beta},$$

where α and β are the solutions of the congruences (121) when $\delta = 1$.

The conditions that U_1^{δ} is found in the group $\{T_1, T_2\}$ reduce to

$$\left.\begin{array}{l} k_2\delta - k_1\alpha \equiv 0, \\ k_3\delta - k_2\alpha \equiv 0. \end{array}\right\} \pmod{\bar{N}}. \tag{123}$$

But the solution of (123) has already been found, since these congruences are identical with (27). If we make $\bar{\mathfrak{S}} = Qr$, this solution is

$$\left.\begin{array}{r} \alpha = \bar{\mathfrak{S}}\,(n + \lambda t), \\ \delta = -\bar{\mathfrak{S}}\,(v_n + \mu t), \\ vk_2 + k_1 \equiv 0 \pmod{t}, \\ v_n \equiv nv \pmod{t}, \end{array}\right\} \tag{124}$$

or

$$\left.\begin{aligned} \alpha &= \bar{\mathfrak{S}}(w_n + \lambda' t), \\ \delta &= -\bar{\mathfrak{S}}(n + \mu' t), \\ wk_1 + k_2 &\equiv 0 \pmod{t}, \\ w_n &\equiv nw \pmod{t}. \end{aligned}\right\} \tag{125}$$

To find the least positive value δ satisfying (123), one may put $n = t - 1$ and $\mu' = 1$ in (125). The value for δ and the corresponding values for α, say α_s and δ_s, are then

$$\begin{aligned} \delta_s &= \bar{\mathfrak{S}}, \\ \alpha_s &= \bar{\mathfrak{S}}(w_{t-1} + \lambda t). \end{aligned} \tag{126}$$

It may be proven that values for λ and β exist both $\overline{\gtrless} N_2$, which will satisfy the conditions for

$$U_1^{\delta} = T_1^{\alpha} T_2^{\beta},$$

where δ_s and α_s are substituted for δ and α respectively. It follows that the order of the group

$$\{T_1, T_2, U_1, U_2\} \text{ is } N_1 N_2 \bar{\mathfrak{S}}.$$

To find the order of the group $\{T_1, T_2, S\}$, we have

$$ST_i^{\alpha} = U_i^{\alpha} S, \qquad i = 1 \text{ or } 2.$$

Let $$SU_i^{\alpha} S^{-1} = V_i^{\alpha},$$

whence $$S^2 T_i^{\alpha} S^{-2} = V_i^{\alpha}$$

and $$S^2 T_i^{\alpha} = V_i^{\alpha} S^2.$$

But $T_i U_i V_i = 1$, since $\Sigma k \equiv 0 \pmod{N_1}$ and $\Sigma k' \equiv 0 \pmod{N_2}$.

Therefore, $$V_i^{\alpha} \equiv T_i^{-\alpha} U_i^{-\alpha}$$

and $$S^2 T_i^{\alpha} = T_i^{-\alpha} U_i^{\alpha} S_1^2.$$

It follows that every element of the group $\{T_1, T_2, S\}$ may be put in the form

$$\begin{aligned} &T_1^{\alpha} T_2^{\beta} U_1^{\gamma} S^{\delta}, \\ &\alpha = 0, 1, 2 \ldots\ldots N_1 - 1, \\ &\beta = 0, 1, 2 \ldots\ldots N_2 - 1, \\ &\gamma = 0, 1 \quad \ldots\ldots \bar{\mathfrak{S}} - 1, \\ &\delta = 0, 1, 2. \end{aligned}$$

Therefore, the order of the group $\{T_1, T_2, S\}$ is

$$3N_1 N_2 \bar{\mathfrak{S}} = 3NQr.$$

§18.—*The Order of the Group* $\{T_1, T_2, s\}$.

If by s we mean the substitution (i, l), and if we put

$$sT_1s = u_{il}, \quad sT_2s^{-1} = v_{il},$$

so that

$$s\,u_{il}\,s^{-1} = T_1, \quad s\,v_{il}\,s^{-1} = T_2,$$

it may be seen that every substitution of the group $\{T_1, T_2, s\}$ may be put in the form

$$T_1^{\alpha}\,T_2^{\beta}\,u_{il}^{\gamma}\,v_{il}^{\delta}\,s^{\epsilon}.$$

We have then to find the lowest powers of u_{il} and v_{il} that occur in the group $\{T_1, T_2\}$.

The group $\{T_1, T_2\}$ contains the substitution v_{il}, for suppose

$$T_1^{\alpha}\,T_2^{\beta} = v_{il}^{\gamma},$$

From this condition

$$\left.\begin{aligned} k_i\alpha + \bar{N}k_i'\beta &\equiv \bar{N}\,k_l'\gamma, \\ k_j\alpha + \bar{N}\,k_j'\beta &\equiv \bar{N}\,k_j\gamma. \end{aligned}\right\} \pmod{N_1}, \tag{127}$$

If $\alpha = \bar{N}_1\alpha_1$, the congruences (127) reduce to

$$\left.\begin{aligned} k_i\alpha_1 + k_i'\beta &\equiv k_l'\gamma, \\ k_j\alpha_1 + k_j'\beta &\equiv k_j'\gamma. \end{aligned}\right\} \pmod{N_2}. \tag{128}$$

The congruences (128) have a solution for any value of γ, since $[(k_i, k_j), N_2] = 1$, hence for $\gamma = 1$.

To find the lowest power of u_{il} contained in $\{T_1, T_3\}$, let

$$T_1^{\alpha}\,T_2^{\beta} = u_{il}^{\gamma}.$$

We have then the two independent congruences

$$\left.\begin{aligned} k_i\alpha - k_l\gamma &\equiv \bar{N}k_i'\beta, \\ k_j\alpha - k_j\gamma &\equiv \bar{N}k_j'\beta. \end{aligned}\right\} \pmod{N_1}. \tag{129}$$

From (129) follow, as necessary conditions,

$$\left.\begin{aligned} k_i\alpha - k_l\gamma &\equiv 0, \\ k_j\alpha - k_j\gamma &\equiv 0. \end{aligned}\right\} \pmod{\bar{N}}. \tag{130}$$

By (65) and (67) we have

$$k_j = q_j\,k_j, \quad k_i - k_l = s't', \quad \bar{N} = QPr't'.$$

With this notation, the solution of (130) is found to be

$$\begin{aligned} \gamma &= Q_j Pr' (n + \lambda q_j t'), \\ \alpha &= Q_j Pr' (-w_n'' + \mu q_j t'), \\ w'' k_i + k_l &\equiv 0 \pmod{q_j t'}, \\ w_n'' &\equiv n w'' \pmod{q_j t'}. \end{aligned}$$

The least value of γ and the corresponding value of α are therefore

$$\begin{aligned} \gamma &= Q_j Pr', \\ \alpha &= Q_j Pr' (t' - w'' + \mu q_j t'): \end{aligned}$$

These values satisfy both the congruences (130), and with them substituted in (129) it may be shown that there exist a set of values for μ and β both $\overline{\lesssim} N_2$, which will satisfy (129). It follows that $\mu_{il}^{Q_j Pr'}$ is the lowest power of u_{il} occurring in the group $\{T_1, T_2, u_{il}, v_{il}\}$, and that the order of this group is $N_1 N_2 Q_j Pr'$.

Every substitution of the group $T_1, T_2, s_{il}\}$ may be put in the form

$$\begin{aligned} & T_1^\alpha T_2^\beta u_{il}^\gamma s_{il}^\delta \\ & \alpha = 0, 1, \ldots . N_1 - 1, \\ & \beta = 0, 1, \ldots . N_2 - 1, \\ & \gamma = 0, 1, \ldots . Q_j Pr, \\ & \delta = 0, 1. \end{aligned}$$

The order of the group $\{T_1, T_2, s_{il}\}$ is therefore

$$2NQ_j Pr' = 2N \frac{\bar{N}}{q_j t'}.$$

§19.—*The Order of the Group* $\{T_1, T_2, s_{i\kappa}, \tau\}$.

The substitutions τ_j and τ_4 are interchangeable with all the substitutions of the group $\{T_1, T_2, s_{i\kappa}\}$, and the order of the group $\{T_1, T_2, s_{i\kappa}, \tau\}$ is therefore $2^2 N \frac{\bar{N}}{q_j t'}$.

If τ is τ_i, let $\theta_\lambda = T_1^\alpha T_2^\beta u_{i\kappa}^\gamma s_{i\kappa}^\delta$

be any substitution of the group $\{T_1, T_2, s_{i\kappa}\}$, and let $s = 2N \frac{\bar{N}}{q_j t'}$. $\tau_i \theta_\lambda$ is either

$\theta_\lambda \tau_i$ or $\theta_\lambda \tau_\kappa$ according as δ is 0 or 1. We may then form the following table:

$$\begin{array}{llll}
\theta_1 = 1, & \theta_2 \quad , & \theta_3, \ldots\ldots & \theta_\rho \quad , \\
\tau_i \quad , & \theta_2\tau_i \quad , & \ldots\ldots & \theta_\rho\tau_i \quad , \\
\tau_l \quad , & \theta_2\tau_l \quad , & \ldots\ldots & \theta_\rho\tau_l \quad , \\
\tau_i\tau_l \quad , & \theta_2\tau_i\tau_l, & \ldots\ldots & \theta_\rho\tau_i\tau_l .
\end{array}$$

If N_1 is even, the first line contains one of the substitutions σ_1, σ_2, σ_3, (51) viz. $T_1^{\frac{N_1}{2}}$.

Suppose $T_1^{\frac{N_1}{2}} = \sigma_i$, then σ_i and $\sigma_l = s_{i\kappa}\sigma_i s_{il}$ and, consequently, $\sigma_j = \sigma_i\sigma_l = \tau_i\tau_l$ are found in the first line of the table. The group is then exhausted by the first two lines of the table. The same argument applies when $T_2^{\frac{N_1}{2}}$.

If $T_1^{\frac{N_1}{2}} = \sigma_j$, the first line contains neither σ_i nor σ_κ, unless N_2 is also even, since $s_{il}\sigma_j s_{il} = \sigma_j$. The second line contains $\sigma_j\tau_i = \tau_l$ and $\tau_i\tau_l = \sigma_j$ is contained n the first line. The group is then exhausted by the first two lines.

If N_1 is odd, the group contains τ_i and $s_{il}\tau_i s_{il} = \tau_l$ which are not found in the first line, and $\tau_i\tau_l = \sigma_j$ which is not found in any one of the first three lines. In this case the order of the group is $2^3 N_1 N_2 \dfrac{\bar{N}}{q_j t'}$.

The same argument holds for $\tau = \tau_l$.

The final result is, the order of the group $\{T_1, T_2, s_{il}, \tau\}$ is $2^2 N \dfrac{\bar{N}}{q_j t'}$, unless N_1 is odd and τ is either τ_i or τ_l, in which cases it is $2^3 N \dfrac{\bar{N}}{q_j t'}$.

§20.—*The Order of the Group* $\{T_1, T_2, S, s\}$.

The group $\{T_1, T_2, S\}$ is a self-conjugate subgroup of the group $\{T_1, T_3, S, s\}$. It follows immediately, since s is of order two, that the order of the latter is twice that of the former.

The order of the group $\{T_1, T_2, S, s\}$ is $2.3.NQr$.

§21.—*The Order of the Group* $\{T_1, T_2, S, s, \tau\}$.

There are two cases:

Case I. $\tau = \tau_4$.

The substitution τ_4 is interchangeable with every substitution of the group $\{T_1, T_2, S, s\}$, hence the order of the group in question is $2^2.3NQr$.

Case II. $\tau \neq \tau_4$.

Subcase 1). If N_1 is even, $\{T_1, T_2, S, s\}$ contains one of the substitutions σ, namely, $T_1^{\frac{N_1}{2}}$ and, consequently, it contains $\sigma_1, \sigma_2, \sigma_3$. Moreover, the substitutions of $\{T_1, T_2, S, s\}$ may be put in the forms

$$T_1^\alpha T_2^\beta U^\gamma S^\delta s^\epsilon \text{ or } T_1^\alpha T_2^\beta U^\gamma \theta_\lambda,$$

where θ_λ is one of the six substitutions of the group $\{S, s\}$. We have

$$\theta_\lambda^{-1}\tau_i\theta_\lambda = \tau_\mu,$$

hence

$$\tau_i\theta_\lambda = \theta_\lambda\tau_\mu.$$

It follows that the substitutions of the group $\{T_1, T_2, S, s, \tau\}$ may all be put in one of the four forms R_ν, $R_\nu\tau_1$, $R_\nu\tau_2$, $R_\nu\tau_3$, where $\nu = 1, 2, 3 \ldots . 2 . 3N_1N_2Q^r$ are the substitutions of the group $\{T_1, T_2, S, s\}$.

But

$$R_\nu = R_\nu\sigma,$$

whence

$$R_\nu\tau_1 = R_\nu\sigma_1\tau_1 = R_\nu\tau_2 = R_\nu\tau_3,$$

so that the group is exhausted by the sets R_{ν_1} and $R_\nu\tau_1$. The order is, therefore, $2^2 . 3N_1N_2Qr$.

Subcase 2). If N_1 is odd, the substitutions R_ν, $R_\nu\tau_1$, $R_\nu\tau_2$, $R_\nu\tau_3$ are all distinct, since otherwise one would have

$$R_{\nu_1}\tau_i = R_{\nu_2}\tau_j,$$

whence

$$R_{\nu_1}^{-1}R_{\nu_2} = \tau_i\tau_j = \sigma_\kappa,$$

but σ_κ cannot occur in the set R_ν. Hence the order of the group is $2^3 . 3NQr$. The result may be stated as follows: If $\tau = \tau_4$, or if $\tau \neq \tau_4$ and N_1 is even, the order of the group $\{T_1, T_2, S, s, \tau\}$ is $2^2 . 3 . NQr$; if $\tau \neq \tau_4$ and N_1 is odd, the order is $2^3 . 3 . N . Qr$.

On the Forms of Unicursal Sextic Scrolls.

By Virgil Snyder.

The study and classification of scrolls whose order is less than or equal to 5 has been, to quite an extent, completed. The following paper will discuss some types of sextic scrolls.

The literature on cubic and quartic scrolls is extensive and well known. The only paper on quintics with which I am familiar is that by H. A. Schwarz, "Ueber die geradlinigen Flächen fünften Grades," in Crelle's Journal, Vol. 67, pp. 23–57. Apart from incidental mention of particular cases, there has been no investigation of sextics. I have used the methods of Cayley, Salmon, and Schwarz, but the new singularities which first appear in the sextic sometimes require a different treatment.

The first section is a direct generalization of the corresponding articles in Schwarz's paper, and I have used his notation throughout; the remaining treatment differs widely in the details from his procedure. In the second section, Schwarz's method of generation is employed in part, and in the third section, the dual of his method.

With the exception of the generators, all plane curves which lie upon a scroll which is not reducible, and which possess the property that only one generator passes through each point, must be of the same genus. This follows from the fact that the generators furnish a one to one correspondence between the points of the two curves.

This principle furnishes a basis of classifying scrolls according to the genus of the curves which can be drawn upon them.

When a straight line which is not a generator, a conic section, a nodal cubic or any unicursal curve lies on a scroll as a simple curve, then every section of the scroll will be unicursal, and the surface itself will be called unicursal, or $p = 0$. Similarly for $p = 1, \ldots.$

For the geometric construction of the various cases, it will be desirable to consider the surface as generated by joining corresponding points of two plane curves of the same genus, or, dually, by the line of intersection of corresponding planes of two developables.

§1.—*General Discussion of the Various Possible Types of Sextic Scrolls.*

A plane passed through any generator g of a sextic scroll will cut from the surface a curve of the fifth order. This curve cuts the generator g in five points. One of these is the point of tangency of the plane through g. In general, each plane is a simple tangent plane, hence the other points are double points on the surface, so that each generator of the scroll is cut by four other generators. There are, therefore, an infinity of planes which pass through two generators. The consideration of these planes furnishes the point of departure of the following discussion.

Every plane through two generators cuts from the surface a quartic curve, which may be reducible. Whenever a plane section of a scroll is reducible, the composite curve of section must consist of a number of generators and an irreducible curve which possesses the property that at least one generator passes through every point, if the surface is not a cone.

Cones will be excluded from the investigation. In case of the sextic scroll the irreducible curve in a plane containing two generators may be a simple or multiple straight line, a conic, a cubic or a quartic curve.

A. If the irreducible curve be a simple straight line, a conic, a nodal cubic, or a trinodal quartic, the surface is of genus 0.

Any scroll whose order is greater than 4, cannot contain more than one simple conic.

B. If the curve of section be a non-singular cubic or a binodal quartic, then $p = 1$.

C. When the curve is a quartic with one node, $p = 2$.

D. When the curve is a quartic without nodes, $p = 3$.

E. It is necessary to make a particular investigation of the case in which the surface contains a multiple linear directrix.

(a.) Suppose the directrix be double. Then every plane through the double line will cut four generators from the surface, because the section of the surface

made by this plane cannot contain any further irreducible part than the double line. Through every point of the double line passes one generator (in which case the directrix is itself a generator) or two. In the first case, the scroll is unicursal. In the second case, the plane which contains the two generators which pass through a point of the double line, does not contain the double line itself. Suppose this were the case. Then every plane through the double line would contain two generators which cut each other on the double line, and two other generators, neither of which can pass through the point of intersection of the first two, because the line was only double.

Hence each of the latter generators must cut the directrix in some other point. If they intersect in different points, then through each such point on the directrix d must pass one other generator of the surface which cannot lie in the plane of the first ones, as the complete section of the surface is accounted for, hence the plane through the last two does not contain d.

The only remaining alternative is that the two latter generators cut d in the same point. This configuration is excluded, as it would not give an irreducible sextic scroll.

Hence, in general, the plane of the two generators does not contain d. Consider the section of the surface made by the plane of the two generators issuing from the same point of d. This plane can contain no other generator, for every generator must cut the double line, hence it would have to pass through the point of intersection of the first two, making the directrix triple, contrary to hypothesis. The curve of section is, therefore, either an irreducible quartic curve or a fourfold line.

In the first case, p cannot be greater than 3. In the latter case, in which the surface has one double and one quadruple directrix, skew to each other, if a plane be passed through any generator, not containing either directrix, it will cut from the surface a quintic curve which has a triple point at the intersection of the generator and the quadruple directrix. If the surface does not contain any double generators, the quintic curve will have no other singular points, hence the curve, and therefore the surface, will have $p = 3$.

For each double generator, the genus of the surface would be reduced by one, hence a (2, 4) scroll may have 3 double generators without becoming reducible.

Every sextic scroll which contains a double generator contains an infinite

number of plane quartic curves whose planes all pass through the double generator.

When the surface is of type (2, 4), these plane curves must all have a double point at the intersection of the double generator and the quadruple directrix—each would then be of genus $p = 2$.

(b.) Suppose the plane of the two generators cuts from the surface a triple line. It will also contain a generator, then every plane through the triple line must cut from the surface three generators, as no other irreducible curve is possible, apart from the multiple directrix. Of these three generators, two may coincide with the directrix, or only one, or none. Being a triple line, three generators will issue from every point. If two of them coincide with the directrix, the triple line counts for simple directrix and double generator, hence the surface is unicursal.

Let two generators issue from each point of the triple line apart from the line itself. There are now two cases to consider, according as the plane of the two generators does or does not contain the triple line. In the first case the plane through the generators would cut from the surface just one more line. This line would lie in the plane π and cut d; through the point in which this extra generator cuts d must pass another generator not lying in the plane, hence the plane of the two latter generators would not contain d. The case, then, in which the two generators lie in a plane with d, can be excluded. The only remaining case is when the plane through the generators which issue from the same point of d does not contain d. The plane will then cut from the surface an irreducible quartic or a cubic and a generator or a conic for particular positions, or, finally, a multiple line, which would have to be a triple line. In this case, d could not be a generator, but 3 generators would lie in the same plane.

The case in which three generators issue from each point of d and all lie in the same plane not containing d, leads also to a new triple line. Every generator must intersect both skew triple directrices. A plane passed through any generator will cut from the scroll a quintic curve having a double point on each directrix and cutting the generator in one more point, the point of contact of the scroll.

The quintic curve has in general no further double points, hence it belongs to the type $p = 4$. As before, the type is reduced by a unit for every double generator, hence a (3, 3) scroll may have 0, 1, 2, 3, or 4 double generators.

(c.) When the irreducible curve contained in a plane passing through two generators is a fourfold line d, four different cases may arise.

From every point of d passes 1, 2, 3, or 4 generators of the surface which are different from d itself.

If only one generator (distinct from d) passes through each point of d, the surface is of genus $p = 0$.

Suppose two generators, distinct from d, pass through every point of d, and let their plane not contain d. This plane will cut from the surface an irreducible quartic, or a multiple directrix.

If the plane of the two generators always passes through the quadruple line, so that the two generators which any plane through d cuts from the surface intersect on d, then, apart from possible multiple generators, no other double lines exist on the surface. This is the (2, 4) case in which the two skew directrices coincide.

If a plane be passed through a generator of a scroll of this type, the section made by it will be a quintic curve having a triple point at the intersection of the generator and the quadruple directrix, and no other multiple point, hence the surface will belong to the category $p = 3$.

Suppose the fourfold line such that three generators, distinct from the line itself, pass through every point of it. Then a plane exists which contains two of them without passing through the fourfold line. This plane will then cut from the surface an irreducible quartic curve or a multiple line; the line is at most a double line.

Suppose, finally, that four generators distinct from the multiple directrix issue from each of its points. If all four lines lie in one plane, the surface is of type (2, 4) and the plane cuts the surface in a double line which is not a generator. This case has already been considered.

(d.) Suppose, finally, that the surface has a fivefold line. Then the plane of two generators contains a quartic curve which has a triple point on the fivefold directrix, hence the surface is unicursal.

It thus appears that sextic scrolls are to be divided into five groups, according as the genus is 0, 1, 2, 3, 4.

§2.—*Unicursal Sextic Scrolls Generated by Two Developables.*

When one variable has been eliminated, the equations of the variable line which generates the scroll is defined as that of intersection of the two planes

$$\left.\begin{aligned} E &\equiv at^m + bt^{m-1} + \dots\dots pt + q = 0, \\ E' &\equiv a't^n + b't^{n-1} + \dots\dots p't + q' = 0, \end{aligned}\right\} m \geqq n,$$

in which $a, b \dots\dots ; a', b' \dots\dots$ are linear homogeneous expressions in x, y, z, w. By eliminating t between these two equations, there results an equation in x, y, z, w of degree $m + n$, which is the order of the surface.

When the two equations $E = 0$, $E' = 0$ do not represent the same plane for any value of t, the eliminant will contain no extraneous factors. Suppose that $E \equiv E'$ when $t = t_0$. Then this plane appears as factor in the scroll, Consider, in this case, a constant $\varkappa$ so determined that $E - \varkappa E' \equiv 0$ when $t = t_0$, which is always possible. $E - \varkappa E' \equiv (t - t_0) E''$. Now, replace $E = 0$ by the plane $E'' = 0$. This process can be continued until no value of t exists for which E, $E^{(n)}$ define the same plane. For unicursal surfaces of the sixth order $m + n = 6$ hence there are three cases to consider:

$$\text{I.} \quad \begin{cases} m = 5 & E \equiv at^5 + bt^4 + ct^3 + dt^2 + et + f = 0, \\ n = 1 & E' \equiv pt \quad + q = 0. \end{cases}$$

$$\text{II.} \quad \begin{cases} m = 4 & E \equiv at^4 \quad + bt^3 + ct^2 + dt + e = 0, \\ n = 2 & E' \equiv pt^2 \quad + qt + r = 0. \end{cases}$$

$$\text{III.} \quad \begin{cases} m = 3 & E \equiv at^3 \quad + bt^2 + ct + d = 0, \\ n = 3 & E' \equiv pt^3 + qt^2 + rt + s = 0. \end{cases}$$

Since the order of a scroll is not changed by duality, it is possible to think of it as generated by joining corresponding points of two twisted curves rather than by the lines of intersection of corresponding planes of two developables, hence

If a (1, 1) *correspondence exist between the points of a straight line and the points of a unicursal quintic, the lines joining corresponding points generate a sextic scroll.*

Similarly for a conic and unicursal quartic, and for two cubics.

[I.] $m = 5, n = 1$.

The equation of the scroll having a quintuple line can be written in the form

$$u_6 + zu_5 + wv_5 = 0, \tag{1}$$

in which u_s, v_s are binary quantics in x, y of order s. The five tangent planes at any point z_1, w_1 of the quintuple line are defined by the equation

$$z_1 u_5 + w_1 v_5 = 0.$$

For certain values of $z_1 : w_1$, 8 in number, two roots of this equation coincide they define the cuspidal generators, which cut the quintuple line in the pinch points. The pinch-points may also be defined as the points in which the quintuple line cuts the octic torse $a\lambda^5 + b\lambda^4 + \ldots\ldots = 0$.

By a suitable linear transformation, equation (1) may be reduced to

$$x^2y^2u_2 + zu_5 + wv_5 = 0.$$

This will be called type I.

A simple mode of generation is obtained from the section made on the scroll by the plane containing two generators. The section is a quartic curve having a triple point. The surface is generated by the lines cutting two such quartics not in the same plane, and the line joining the triple points. Various subforms exist, when the pinch-points coincide. When $z_1u_5 + w_1v_5 = 0$ has a cubic factor two pinch-points coincide, which requires that the octic torse touch the quintuple directrix.

When u_5, v_5 contain a common factor, so that the equation may be written

$$x^2y^2u_2 + (ax + by)(zu_4 + wv_4) = 0,$$

in one position of the generator it coincides with the directrix, $ax + by = 0$ being the osculating tangent plane. This is type II. Subforms exist as under I.

III. $x^2y^2u_2 + (ax + by)(cx + dy)(zu_3 + wv_3) = 0$, two such limiting generators and osculating planes.

IV. $x^2y^2u_2 + u_1 \cdot v_1 \cdot \bar{u}_1 (z\bar{u}_2 + wv_2) = 0$, three such generators.

V. Four such generators.

VI. $x^2y^2u_2 + u_1^2(zu_3 + wv_3) = 0$. In this case two of the five sheets through the quintuple line unite in forming a single cuspidal edge.

VII. $x^2y^2u_2 + u_1^2 \cdot v_1 (z\bar{u}_2 + wv_2) = 0$, cuspidal edge and limiting generator.

VIII. $x^2y^2u_2 + u_1^3 (z\bar{u} + wv_2) = 0$, three sheets in cuspidal edge, i. e. three sheets have common tangent plane.

IX. $x^2y^2u_2 + u_1^3 . v_1 (zu_2 + w\bar{v}_1) = 0.$

X. $x^2y^2u_2 + u_1^2v_1^2 (zu_1 + wv_1) = 0.$

XI. $x^2y^2u_2 + u_1^4 (z\bar{u}_1 + wv_1) = 0.$

In forms V, IX, X and XI, the quintuple line is also fourfold generator, so that only one other generator can issue from each point of the directrix. These scrolls belong to special linear congruences. Their asymptotic lines are algebraic and of order 9.

When in I, u_2 vanishes identically, the five generators which issue from the same point of the quintuple directrix lie in one plane, which also contains a simple linear directrix $z = 0$, $w = 0$, skew to the first one. This is type XII. The surface is contained in a general linear congruence; its asymptotic lines are of order 10. The forms V, IX, X, XI may also be regarded as limiting cases of XII, when the two directrices approach coincidence.

Among the subforms of XII, two varieties are of particular interest, those which are transformed into themselves by a cyclic collineation of order 5 and those whose asymptotic lines are reducible. The scroll

$$zx\left(8x^2 - (3 - \surd 5)\,y^2\right)\left(32x^2 - 5\,(7 + 3\surd 5)\,y^2\right)$$
$$= wy\left(8y^2 - (3 - \surd 5x^2)(32y^2 - 5\,(7 + 3\surd 5)\,x^2\right)$$

is of the first kind.*

In the scroll
$$zx^5 + wy^5 = 0$$

there are two real fourfold pinch-points, at which all five generators coincide; from the other points of the multiple directrix issues but one real generator.

The surface $zx(3x^4 + 5y^4) + wy(5x^4 + 3y^4) = 0$ differs from the preceding by having four double pinch-points, all real, at each of which three generators coincide.

The surface $zx(3x^4 - 5y^4) + wy(5x^4 - 3y^4) = 0$ has four double pinch-points, all imaginary. From every point of the multiple directrix issue three real and distinct generators. The form $zx^3(x^2 - 5y^2) + wy^3(5x^2 - y^2) = 0$ has also three generators issuing from each point, with real double pinch-points. The form $zx^3(x^2 + 5y^2) + wy^3(5x^2 + y^2) = 0$ has two real and two imaginary double pinch-points. In all these forms, the asymptotic lines are of order 5. The necessary

* Ameseder, in the Wiener Berichte for 1890, treats of this kind of cyclic collineation. It is a projection of an axial rotation through 72°.

condition is that at every pinch-point three generators coincide, so that all of the pinch-points coincide in pairs.*

[II.] $m=4$, $n=2$.

By writing

$$\begin{aligned}
&p\,[p^2d-r\,(bp-aq)]-q\,[p\,(cp-ar)-q\,(bq-aq)]\equiv\phi_1,\\
&p^3e-r\,[p\,(cp-ar)-q\,(bp-aq)]\equiv\phi_2,\\
&p^2qe-r\,[p^2d-r\,(bp-aq)]\equiv\phi_3,\\
&pq\,(qe-dr)+r\,[r\,(cp-ar)-p^2e]\equiv\phi_4,\\
&q\,[r\,(cr-ep)-q\,(dr-qe)]-r\,[r^2b-p\,(dr-qe)]\equiv\phi_5,
\end{aligned}$$

the results of partial elimination of t may be written

$$\phi_1t+\phi_2=0,\quad \phi_2t+\phi_3=0,\quad \phi_3t+\phi_4=0,\quad \phi_4t+\phi_5=0,$$

and the equation of the surface may be written in either of the following forms:

$$\begin{aligned}
&\phi_2^2-\phi_1\phi_3=0,\quad \phi_1\phi_4-\phi_2\phi_3=0,\quad \phi_3^2-\phi_2\phi_4=0,\\
&\phi_1\phi_5-\phi_2\phi_4=0,\quad \phi_2\phi_5-\phi_3\phi_4=0,\quad \phi_4^2-\phi_3\phi_5=0.
\end{aligned}$$

Each of these equations contains an extraneous factor; these are p^2, pq, pr, q^2-pr, qr, r^2 respectively.

These expressions are not all independent. The relation

$$r\phi_1-q\phi_2+p\phi_3\equiv 0$$

can be easily verified. By equating the values of t,

$$\frac{\phi_1}{\phi_2}=\frac{\phi_2}{\phi_3}=\frac{\phi_3}{\phi_4}=\frac{\phi_4}{\phi_5}=\frac{r\phi_1-q\phi_2+p\phi_3}{r\phi_2-q\phi_3+p\phi_4}, \tag{2}$$

hence, $$r\phi_2-q\phi_3+p\phi_4\equiv 0,$$

and similarly, $$r\phi_3-q\phi_4+p\phi_5\equiv 0.$$

The surfaces ϕ_1, ϕ_2 do not intersect in a plane curve, hence ϕ_3 passes through their curve of intersection; similarly for ϕ_4, ϕ_5, so that all the surfaces pass through the same curve. Again, from the form of equations (2), the non-reducible part of

† A similar case of a (3, 1) scroll was discussed by me in the American Journal, Vol. 22, p. 257. The algebraic condition is discussed in Salmon's "Higher Algebra," 4th ed., p. 162. The Jacobian of u_3, v_3 must be a perfect square.

this curve is a double line on the sextic scroll. The surfaces ϕ_2, ϕ_3 have the four lines p, r; p, a; e, r; p, q in common; p, r is a double line on ϕ_3, and the two surfaces touch along the line p, a. The plane $r = 0$ touches ϕ_2 the whole length of p, r and is an inflexional plane. The total curve of intersection of ϕ_2, ϕ_3 is of order 16; the common lines account for a curve of order 6, hence the double curve on the scroll is of order 10. The point p, q, r is a threefold point on ϕ_2 and on ϕ_3, hence a ninefold point on their curve of intersection. The line p, r being a single line on ϕ_2 and a double line on ϕ_3, has a double point at p, r, q. Similarly, the line q, r has three points at the ninefold point; no other lines common to the two surfaces pass through the multiple point, hence the curve has a fourfold point.

The unicursal sextic scroll of form [II] has a double curve of order 10, *which has a fourfold point.*

This scroll will be called type XIII.

The surface $\phi_2^2 - \phi_1\phi_3 = 0$ is the envelope of the quadratic pencil of quartic surfaces

$$\phi_1 t^2 + 2\phi_2 t + \phi_3 = 0;$$

each surface of the pencil passes through the double curve; the residual curve is then of order 4. As the surface must always touch the scroll, the two must either touch along one generator and intersect in two others or touch along two.

The same may be said of the two other pencils,

$$\phi_2 t^2 + 2\phi_3 t + \phi_4 = 0,$$
$$\phi_3 t^2 + 2\phi_4 t + \phi_5 = 0.$$

Every generator cuts the double curve in 4 points, hence the curve cannot lie on a quadric or cubic scroll, for in that case every generator of the sextic would also belong to the simpler surface. Through every point of the double curve pass two generators, each of which cuts the double line in three other points. The cone of order 9 having the double curve for directrix and any point upon it for vertex, contains two triple and one fourfold generator, the latter passing through the fourfold points.

A non-reducible sextic scroll cannot have a double curve of order 10 with a double point of order higher than the fourth, for every line joining the multiple point to any other point of the curve would be a part of the surface, and this cone would be of order less than 6, hence the surface would be degraded.

This reasoning cannot be applied to cases in which the double curve of order 10 is itself reducible.

Some of the possible forms into which the double curve can break up will now be considered.

Let the surface have a fourfold directrix line. If $p_1 = 0$, $q_1 = 0$ be the equations of two planes through the line, then every other plane through the line can be represented by an equation of the form $p_1\lambda + q_1 = 0$.

Any plane through the directrix will cut two generators from the surface, which are generally distinct from the directrix. To every value of λ correspond therefore two values of t; to every value of t corresponds but one plane through the directrix, hence to one value of λ. λ is therefore a rational quadratic function of t.

By proper linear substitution this can always be brought to the form

$$\lambda = t^2, \quad \text{or} \quad pt^2 + q = 0,$$

which, associated with

$$at^4 + bt^3 + ct^2 + dt + e = 0,$$

defines the scroll. The equation of the surface is therefore

$$[p(qc - pe) - q^2a]^2 + pq(pd - bq)^2 = 0. \tag{3}$$

This method must be modified somewhat when one of the four generators which issue from each point of the multiple directrix coincides with the directrix itself In that case the equation $at^4 + bt^3 + ct^2 + dt + e = 0$ is expressible in the form

$$(a't^3 + b't^2 + c't + d')(t - t_0) + \alpha p + \beta q = 0.$$

Corresponding restrictions must be made when two or more generators coincide with the multiple directrix.

From equation (3), when $p = 0$, $q^4a^2 = 0$; when $q = 0$, $p^4e^2 = 0$, hence the planes p, q are both torsal. Besides the fourfold line p, q, the surface has as double curve the line of intersection of the cubic scroll $p(qc - pe) - q^2a = 0$ and the quadric $pd - bq = 0$. It is a quartic curve of the second kind; the generators of one system of the hyperboloid $pd - bq = 0$ meet it in three points, those in which they meet the cubic scroll. Those generators of the other system cut the curve but once, as they cut the double line which is a generator of the first system. The quartic cuts the fourfold line in three points. This is type XIV.

This quartic curve may now break up into a cubic and a straight line, into a

conic and two straight lines, or finally, into four straight lines. (A quartic of the second kind cannot be composed of two conics.)

When the quartic curve breaks up into a cubic and a straight line, it is necessary that the quadric has a second generator in common with the cubic scroll. Let this common generator be cut from the plane $p\lambda_0 + q = 0$. Then the plane $\lambda_0^2 a + \lambda_0 c + e = 0$ is identical with $\varkappa_0 (b\lambda_0 + d) + \mu_0 (p\lambda_0 + q) = 0$. Under the same hypothesis, the following identity also exists:

$$p(qc - pe) - q^2 a \equiv p\varkappa_0 (bq - pd) + (\lambda_0 ap + cp - bp\varkappa_0 - \mu_0 p^2 - qa)(p\lambda_0 + q),$$

which shows that the cubic curve lies on the quadrics $pd - bq = 0$, $\lambda_0 ap + cp - \varkappa_0 bp - \mu_0 p^2 - qa = 0$. It cuts the line p, q in two points. The double line $p\lambda_0 + q$, $b\lambda_0 + d$ cuts p, q in one point. The three lines together constitute the double curve of order 10. The cubic curve cuts the line $p\lambda_0 + q$, $b\lambda_0 + d$ in one point. If

$$\lambda_0 ap + cp - \varkappa_0 bp - \mu_0 p^2 - qa$$

be denoted by u, the equation of the surface becomes

$$\{\varkappa_0 p (bq - pd) + u (p\lambda_0 + q)\}^2 + pq (pd - bq)^2 = 0.$$

It is type XV. The second double line is a double generator of the surface.

The cubic curve may further break up into a conic and a straight line. In this case the two quadrics

$$u = 0, \quad qb - pd = 0$$

have a generator of each system in common; the second generator lies in the plane $p\lambda_0' + q = 0$; then

$$(\lambda_0 a + c - \varkappa_0 b - \mu_0 p) + \lambda_0' a \equiv \varkappa_0' (p\lambda_0' + q) + \mu_0' (b\lambda_0' + d),$$

which may be rearranged in the form

$$(\lambda_0 a + c - \varkappa_0 b - \mu_0 p) p - aq \equiv \mu_0' (pd - qb) - (a - \varkappa_0' p - \mu_0' b)(p\lambda_0' + q).$$

The second double line is $p\lambda_0' + q$, $b\lambda_0' + d$, and the conic is defined by

$$pd - qb = 0, \quad a - \varkappa_0' p - \mu_0' q = 0.$$

The configuration of multiple lines now consists of the fourfold line p, q; two double lines, each cutting p, q and skew to each other, and a conic which

cuts each of these three lines once. The equation of the surface is

$$\{[\varkappa_0 p + \mu_0'(p\lambda_0 + q)](pd - qb) - (a - \varkappa_0' p - \mu_0' b)(p\lambda_0' + q)(p\lambda_0 + q)\}^2 + pq\,(pd - bq)^2 = 0\,.$$

This is type XVI.

The two double lines are double generators.

If the two surfaces

$$u = 0\,, \quad qb - pd = 0$$

intersect in four lines, two of each system, the two lines which cut p, q will be double generators, and that in the same system as p, q will be a double directrix. The surface will then have a fourfold directrix, a double directrix skew to it, and three double generators. The plane $a - \varkappa_0' p - \mu_0' q = 0$ can now be written

$$v \equiv \alpha\,(p\lambda_1 + q) + \beta\,(b\lambda_1 + d) = 0\,,$$

wherein $p\lambda_1 + q$, $b\lambda_1 + d$ is one of the new double generators; the other lies in the plane

$$w \equiv \alpha\,(p\lambda_0' + q) + \beta\,(b\lambda_0' + d) = 0\,,$$

since

$$\frac{p}{b} = \frac{q}{d} = \frac{p\lambda_1 + q}{b\lambda_1 + q} = \frac{-\beta}{\alpha} = \frac{p\lambda_0' + q}{b\lambda_0' + d}\,.$$

The first line is a generator, the other the double directrix. If $p\lambda_0' + q \equiv p'$, $p\lambda_1 + q \equiv q'$, and since

$$pd - qb = \frac{1}{\beta\,(\lambda_1 - \lambda_0)}\,[v\,(p\lambda_0' + q) - w\,(p\lambda_1 + q)] \equiv m\,(vp' - wq')\,,$$

the equation of the surface may be written

$$\{(Ap' + Bq')(vp' - wq') - vp'\,(Cp' + Dq')\}^2 + (vp' - wq')^2(Ep'^2 + Fp'q' + Gq'^2) = 0\,.$$

The line p', q' is fourfold directrix, v, w double directrix, and q', v; w, p'; $b\lambda_0 + d$, $p'(\lambda_0 + \lambda_1) - q'(\lambda_0 + \lambda_1')$ are double generators. This is type XVII. Several subtypes exist when the pinch-points coincide in various ways.

The two quadrics, u, $pd - qb$, may touch each other along p, q; in this case, $\beta = 0$ in the expression for v, and since $w \equiv \dfrac{p'}{\alpha}$, the equation of the surface becomes

$$[f_1(p, q)(pd - qb) - f_3(p, q)]^2 = pq\,(pd - qb)^2.$$

In this case the surface has a fourfold directrix, a double generator coinciding with it, and three other double generators. This is a limiting case of the last type when the two directrices approach coincidence. It is type XVIII. If the equation be developed, the terms

$$[(\kappa_0 + \lambda_0\mu_0')p^2 + 2(\mu_0'(\kappa_0 + \lambda_0\mu_0') + 1)pq + \mu_0'^2q^2][pd - qb]^2$$

are infinitesimal of order 4, the other terms are of order 5 or 6 when $p = 0$, $q = 0$. Two planes, $pg + q = 0$, $pg' + q = 0$, are tangent along p, q. Each cuts a fivefold line from the surface. The other two tangent planes are coincident with each other and with the tangent planes of $pd - qb = 0$ along p, q. The surface has self-contact along this line. Every plane through p, q cuts two generators from the surface, which intersect on p, q. If one sheet of the surface be described by continuous motion of each line, the two sheets will pass through each other at the double generators and touch each other along p, q.

Now, consider the case in which the sextic scroll has a double rectilinear directrix. This requires that all of the planes of the form $a\lambda^4 + \ldots.$ belong to the same axial pencil. The equation can be derived from the general one by writing $a + \kappa_1 b$, $a + \kappa_2 b$, $a + \kappa_3 b$ for c, d, e.

The remaining part of the double curve is of order 9; it has, as in the general case, a fourfold point and lies on a cone of order 5 which has the fourfold point at its vertex. The curve cuts the double directrix in 3 points. Every plane through the double directrix cuts four generators from the surface; these generators intersect in the 6 points of the double curve which lie in the plane, not on the double straight line. This is type XIX. All the generators touch the cone $q^2 - 4pr = 0$, hence the surface is contained in the quadratic congruence defined by the special quadratic complex formed by the tangents to this cone, and the special linear complex whose axis is the line a, b. The two points in which the line a, b cuts the cone are pinch-points or points of intersection of double generators.

If the coordinates of one of these points of intersection cause the t discriminant of the quartic pencil $at^4 + \ldots.$ to vanish, the generator of the quadric cone which issues from that point will be a double generator of the sextic scroll. The configuration of double lines consists of the double directrix, a double curve of order 8 cutting the directrix in two points and having a triple point and finally a double generator passing through the triple point. This is type XX.

If the coordinates of both points of intersection cause the t discriminant to vanish, both generators issuing from these points will be double generators of the scroll. The double curve is now of order 7, has a double point and cuts the directrix in one point. This is type XXI.

Three double generators cannot appear unless the residual curve consists of a fourfold line; this case was noticed as type XVII. As a subform of XX, the two pinch-points may unite without giving rise to a double generator; i. e. the line a, b may touch the quadric cone.

[III.] $m = 3$, $n = 3$.

By writing

$$\begin{aligned}(as - pd)(pc - ar) - (pb - aq)(sb - qd) &\equiv \psi_1,\\ (rd - cs)(pb - aq) - (as - pd)^2 &\equiv \psi_2,\\ (pd - as)(sb - dq) - (cs - rd)(pc - ar) &\equiv \psi_3,\end{aligned}$$

the equation of the surface is expressed in the form

$$\psi_1 \psi_3 - \psi_2^2 = 0,$$

which has the extraneous factor $pd - as$.

From the identical relation

$$(as - pd)\,\psi_1 + (pc - ar)\,\psi_2 + (aq - bp) \equiv 0,$$

it is seen that ψ_1, ψ_2, ψ_3 pass through the same curve, which is a double curve on the sextic scroll.

ψ_1, ψ_2 intersect in the cubic $pd - as$, $pb - aq$ and the line s, d; they touch along the line a, p.

ψ_2, ψ_3 intersect in the cubic $pd - as$, $cs - rd$ and the line a, p; they touch along the line d, s.

ψ_1, ψ_3 intersect in the quartic $sb - dq$, $pc - ar$ and the two lines p, a; d, s.

The residual curve of order 10 is the double curve of the sextic scroll. The lines a, p; d, s are simple generators of the scroll. This curve has no fourfold points, but has, in general, four triple points. (Salmon's "Algebra," lesson 18.) This scroll is type XXII.

Next, consider the scroll with a triple directrix. Its equation can be derived from the last type by writing $c = a + \varkappa_1 b$, $d = a + \varkappa_2 b$, since all the planes of

the system $at^2 + bt^2 + \ldots\ldots = 0$ belong to the same axial pencil. The residual curve is of order 7. Every plane through the triple directrix cuts three generators from the surface which do not in general intersect on the triple line, hence the triple directrix cuts the double curve in four points. This is type XXIII.

The four points in which the line a, b cuts the quartic developable $pt^3 + \ldots\ldots = 0$ are pinch-points; when the coordinates of one of these points causes the discriminant of $at^3 + \ldots\ldots = 0$ to vanish, the scroll has a double generator which passes through a singular point of the double curve of order 6. This is type XXIV.

In the same manner, the surface may have two or four double generators, or a double and a triple generator. These are types XXV, XXVI, XXVII. The last five types belong to the congruence formed by the tangents to the quartic developable which cut a fixed line. In type XXVI, the double curve of the scroll is of order 3.

From every point of the triple line a, b issue three generators. Let ϖ be a plane containing two generators g, g' issuing from a point of a, b. The plane will cut from the surface a quartic curve having four double points and passing through the intersection with a, b. The curve cannot consist of two conics, for a sextic scroll can have but one, hence the section must consist of straight lines. The one intersecting a, b is a generator and the others cannot be, hence they must all coincide. The double lines consist of two triple directrices, skew to each other, and four double generators. The equation of this surface can be most easily derived from that of type XXIII by putting $r = p + \lambda_1 q$, $s = p + \lambda_2 q$ in that equation.

A large number of subtypes can be derived from this type by having the pinch-points coincide and by the appearance of a triple generator.

Finally, the two skew directrices may coincide. The three generators which issue from each point of a, b lie in a plane passing through a, b. There are four pinch-points on the line a, b. This is type XXVIII. It may have a triple generator, type XXIX.

§3.—*Scrolls Generated by Two Curves.*

Sextic scrolls will now be studied from the dual standpoint, the locus of a line joining corresponding points of two curves.

Let $\xi_i = \xi_i(\lambda)$, $\eta_i = \eta_i(\mu)$, $i = 1, 2, 3, 4$ be the equations of two unicursal curves, the parameters λ, μ being related by the equation

$$f(\lambda, \mu) = 0.$$

Then
$$x_i = \xi_i(\lambda) + \sigma\eta_i(\mu)$$

will define a scroll which is unicursal when the function $f(\lambda, \mu) = 0$ is so. If ξ, η be of order m, n and $f(\lambda, \mu) = 0$ be of degree m_1 in λ, n_1 in μ, ξ will be an n_1-fold curve on the scroll, η will be an m_1-fold line; the scroll will be of order

$$mn_1 + m_1 n,$$

but the order will be reduced by unity for every point of intersection of ξ, η, which is a self-corresponding point. When f is unicursal, the parameters λ, μ, σ can be rationally eliminated by a process analogous to that employed for unicursal curves.

A scroll having a simple rectilinear directrix must be unicursal; every plane which contains more than one generator must contain the directrix, and hence $n - 2$ other generators. No simple plane curve of order lower than $n - 1$ can exist on the surface.

The double curve is of order $\frac{1}{2}(n-1)(n-2)$, and it cannot contain double generators as a component part except as one or more generators may coincide with the simple directrix.

When the directrix δ is a simple line on the surface, it cannot be cut by the double curve. The scroll can be generated by joining corresponding points of the line and a unicursal curve c of order $n - 1$ whose plane does not contain δ if it be a plane curve. There are $n - 1$ fundamental types, according as δ passes through a $\varkappa$-fold point on the curve; $\varkappa = 0, 1, \ldots. n - 2$. When δ passes through a $\varkappa$-fold point on c, $\varkappa$ generators coincide with δ and any plane through δ will contain but $n - \varkappa - 1$ generators. When $\varkappa = 1$, the double curve intersects δ in $n - 3$ points; in general, in $\varkappa(n - 2 - \varkappa)$ points. The line δ now counts as a nodal curve of order $\frac{\varkappa}{2}(\varkappa + 1) + r$; the residual curve is of order

$$\tfrac{1}{2}(n-1)(n-2) - \frac{\varkappa}{2}(\varkappa + 1) - r,$$

r being the number of simple coincident tangents of c at points of intersection

with δ. Each generator cuts the residual curve in $n - x - 2$ points. In particular, if $x = n - 2$, the $n - 1$-fold line is the only nodal line, and the equation of the scroll is of the form

$$u_n(x, y) + u_{n-1}(x, y)z + v_{n-1}(x, y)w = 0,$$

in which u_{n-1}, v_{n-1} have a common factor of order $n - 2$. In case the curve c has an $n - 2$-fold point and does not cut δ, a special form is one having an $n - 2$-fold line and a simple line

$$xf_{n-1}(x, w) + y\phi_{n-1}(z, w) = 0.$$

In no other case can the nodal curve have a straight line other than δ for a component. These scrolls all belong to a special linear complex; in the two cases just mentioned they are contained in a linear congruence, the former being special, the latter general. The theory will now be applied when $n = 6$.

§4.—*Dual of* (1, 5) *Types.*

When the line δ does not cut the quintic curve, every plane through δ will cut five lines from the surface which intersect in 10 points not lying on δ. They may all coincide, giving type XII, already considered. If the quintic curve does not have a fourfold point, this case is excluded. A double conic, cubic, quartic or quintic is easily proven not to belong to the nodal curve, hence: *When a sextic scroll contains a simple rectilinear directrix which is not a generator, and a quintic curve without a fourfold point, then the nodal curve cannot be reducible.* This c_{10} must accordingly have ∞^1 four-point secants. There are two families according as the nodal curve has a fourfold point or four threefold points. A similar theorem exists for every scroll of even degree $2m$. The nodal curve is of order $(2m-1)(m-1)$, and contains ∞^1 secants, each of which cuts it in $2(m-1)$ points. Curves exist having an infinite family of secants which cut the curve in more than $2(m-1)$ points, but in that case only one such secant can be drawn through each point of the curve. Type XXX.

When δ cuts the quintic curve c_5 once, the residual curve is of order 9. Type XXXI. The curve may now be composite; its factors are a sextic and a cubic. The scroll may be generated by a (1, 2) correspondence between points on δ and a cubic which cuts δ once, the point of intersection being self-corresponding. The cubic may be twisted, type XXXII.

A different form exists when the cubic is plane. Type XXXIII. A simple

illustration is furnished by joining corresponding points of the line $x=0, y=0, z=\mu$ and the curve $x=\lambda^3$, $y=\lambda$, $z=\lambda=\mu(\mu-1)$.

The residual curve is a sextic which cuts δ twice, and is cut twice by every generator.

The sextic curve may break up into two cubics, so that the nodal curve consists of δ and three (twisted) cubics. The equations may now be written by joining corresponding points of two twisted cubics which intersect in five points; the correspondence being a unicursal (2, 2) such that four points are self-corresponding and the fifth the double element of the (2, 2) correspondence. This is type XXXIV.

No other components of the residual nodal curve can exist.

When δ passes through a node on c_5, δ is a triple line and the residual of order 7. This is type XXXV.

A special form exists when the sections containing a generator cut from the scroll a quintic having a tacnode at the trace of δ; in this case δ counts for a fourfold line (i. e. equivalent to a nodal quartic) and the residue is of order 6. This is type XXXVI.

The sextic may break up into two (twisted) cubics, XXXVII, or finally, a triple conic, as is shown by the scroll

$$(y^3+x^2z)^2=x^3yw^2.$$

This is type XXXVIII. The last form is generated by joining corresponding points of $x=0$, $y=0$ and $x^2=y^5$, $z=0$ by a (1, 1) correspondence.

By applying Clebsch's method* for finding the asymptotic lines of the last scroll, they are found to be

$$x=\lambda^6, \quad y=\lambda^3, \quad z=20+c\lambda^4, \quad w=21\lambda+c\lambda^5.$$

They all have a common osculating plane at the pinch-point (0, 0, 1, 0). Any plane through δ cuts each curve in three points besides the pinch-point on δ, one on each generator in the given plane. Two of the generators are always imaginary. The scroll is contained in the special linear complex $p_{12}=0$ and in the special quadratic complex $p_{12}^2+p_{14}\cdot p_{34}=0$, to which the axis δ belongs.

When δ cuts c_5 in a triple point, the residual is a quartic of the second kind and δ counts for a sextic. This is type XXXIX.

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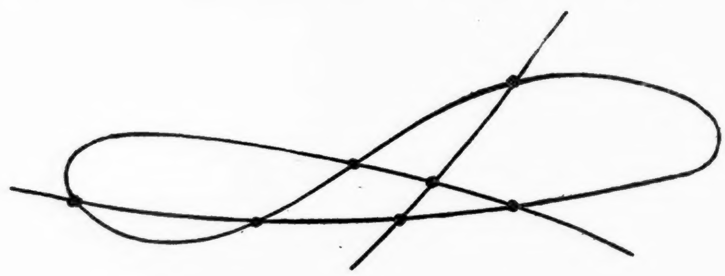
* "Ueber die Haupttangentencurven bei windschiefen Flächen," Crelle, Vol. 68, p. 151.

For particular configurations of the triple point on c_5, the intersection may count for 7, leaving only a nodal cubic. This is type XL.

For example, in the scroll

$$(y^3 - x^2w)^2 = xy\,(xw - xz - y^2)^2,$$

a plane through δ will cut a sextic curve having a singularity of the form



he equations of the odal cubic are $x = \lambda^3$, $y = \lambda^2$, $z = 1 - \lambda$.

The cubic may become plane, type XLI.

E. g., the scroll

$$(y^3 + x^2z)^2 = x^3yw^2$$

has x, y for a fourfold line and $w = 0$, $y^3 = x^2z$ for a nodal cubic curve. The asymptotic lines are

$$x = 3\lambda^6, \quad y = 3\lambda^4, \quad z = 5 - c\lambda^4\sqrt{\lambda}, \quad w = 8\lambda - c\lambda^5\sqrt{\lambda}.$$

They all have four-point contact at the pinch-point (0, 0, 1, 0).

§5.—*Dual of* (2, 4) *Typ* .

The scrolls, which are the dual of type (2, 4), can be generated by joining corresponding points of a conic section and a unicursal quartic curve. The plane of the conic contains four generators, one through each of the four points in which it cuts the quartic. The conic intersects the nodal curve, which is of order 10, in four points. This is type XLII. The constants may be easily arranged so that the plane of the conic contains a double generator, type XLIII, or two double generators, XLIV, without the residual nodal curve being composite. No other multiple generators can exist, nor any simple plane curve, except the conic, of order less than 4.

In general, no sextic scroll except those contained in a linear congruence

with distinct directrices can have more than two distinct double generators, for, if there be three, the semi-quadric determined by them could intersect the scroll nowhere else; the complete intersection of the quadric and the scroll is of order 12, hence the residual is made up of lines of the other semi-quadric.

Specializations of the forms of the directrices are:

(α) a double line and a quartic;
(β) a conic and a double conic;
(γ) a conic and a fourfold line;
(δ) a twofold line and a double conic;
(ε) a twofold line and a fourfold line, already considered.

(α.) A double line and a unicursal quartic generate a sextic scroll having a double directrix and a residual nodal curve of order 9; there may be one or two double generators. These types have already been considered (XIX, XX, XXI).

When δ cuts c_4, it counts for a triple line, and the residual curve is of order 7; this is type XLV, it is distinct from XXIII, as the line δ was triple directrix there may be one double generator, type XLVI, or two, type XLVII; the residual curve must cut δ in three points.

When δ cuts c_4 in a double point, it counts for a nodal curve of order 6, and the residue is a quartic cutting δ three times; type XLVIII. Each generator cuts the quartic once. The configuration of double curves is the same as in XIV, but the types are essentially distinct, as δ is a double generator. There may be a double generator, XLIX. The residue is now a cubic or two double generators and a conic, cutting δ once. Type L.

The last two are distinct types (compare XV, XVI). When δ passes through a triple point on c_4, there can be no other nodal line; δ now counts for a double director and a triple generator. It always belongs to one of the types I to XII.

(β.) A scroll, generated by a (1, 2) correspondence between two conics, has a conic and an octic for nodal curves, LI; there may be one double generator, type LII, or two, type LIII; lines of higher multiplicity may enter. E. g., consider the conics

$$z^2 - xy = 0, \quad w = 0; \quad zw = y^2, \quad x = 0,$$

or

$$x = 0, \quad y = \lambda^2, \quad z = 1, \quad w = \lambda^4,$$
$$x = \lambda^2, \quad y = 1, \quad z = \lambda, \quad w = 0.$$

Then the scroll becomes

$$(w+x)^4z^2 - 2zwy^2(w+x)^2 + w^2y^4 = (w+x)^3x^2y.$$

The line $w+x=0$, $y=0$ is a fourfold directrix and not a generator. The line $w=0$, $x=0$ is a double line which is cut by any plane in a tacnode of the curve of section. This is type LIV. It differs from XVI by having a tacnodal generator.

(γ.) A scroll may be generated by joining points on a conic to a fourfold line by a (1, 1) correspondence. This type has already been considered. Every scroll having a fourfold line contains either a double line or a conic section. If the line cut the conic, it is a fourfold line and simple generator; there is no other nodal line on the scroll.

(δ). A sextic scroll may be generated by a unicursal (2, 2) correspondence between a straight line and a conic. The residual curve is of order 7, type LV. There may be one double generator and a residual curve of order 6, type LVI. In case the line intersects the conic, it is also a double generator; the residual must consist of two double generators, as four intersections of an arbitrary generator and the nodal curve are already accounted for. This type was found before (XVI).

§6.—*Scrolls having Two Double Conics.*

When two conics, which lie in different planes but which have two points of intersection, are put in (2, 2) correspondence in such a way that each point of intersection is a single self-corresponding point, the lines joining corresponding points will be a sextic scroll. Let the conics be

$$x=\lambda,\quad y=\lambda^2,\quad z=0,\quad w=1,$$
$$x=0,\quad y=\mu^2,\quad z=\mu,\quad w=1.$$

The general form of the correspondence is

$$a\lambda^2\mu + b\lambda\mu^2 + c\lambda^2 + d\lambda\mu + e\mu^2 + f\lambda + g\mu = 0.$$

The equation of the scroll can be written in the form

$$\begin{vmatrix} bz-ax-cw & (e+c)z-by-dx & f & ey+gx & 0 \\ 0 & bz-ax & c & by+fw+dx-ez & ey+gx-fz \\ wz & x^2-z^2 & y & 0 & 0 \\ 0 & w & 1 & -z & 0 \\ 0 & 0 & z & x^2-wy & zy \end{vmatrix} = 0.$$

The residual curve is of order 5 and is cut by every generator in two points. When the surface is unicursal, a double generator exists. The latter is expressed by a double point not at (0, 0) nor at (∞, ∞) on the cubic curve in λ, μ. The plane $z = 0$ contains the two generators $ax + cw = 0$, $ey + gx = 0$. The plane $x = 0$ contains the generators $cy + fz = 0$, $ew + bz = 0$. This is type LVII.

When
$$\frac{f}{g} = \frac{b}{a} \text{ and } c = e$$
there is a third double conic lying in the plane $gx - fz = 0$ and passing through the points $(0, 0, 0, 1)$, $(0, 1, 0, 0)$. The residual is now a twisted cubic which meets each generator once. Type LVIII. If the new conic touches the plane $y = 0$ or $w = 0$, the scroll reduces to a quintic and the common tangent plane.

If
$$c + e = 0, \quad f = 0, \quad b + ad = 0,$$
the conic is replaced by a double rectilinear directrix and a double generator. There is no other double generator. The equations of the directrix are
$$ax = a^2 y + (e + ag) w, \quad ax + adz = ew.$$
The residual is a twisted quartic which cuts every generator once. Type LIX. A particular case of these types is when the two conics touch each other; the surface mentioned by de la Gournerie* as a factor of a composite form of the "quadrispinale" of order 8 is the result.

If $c = e = 0$, the double generator is the line joining the points of intersection of the two conics. If, in this case, $\dfrac{f}{g} = \dfrac{b}{a}$, the third double conic breaks up into the double generator and the double rectilinear directrix
$$bdz + y + bgw = 0, \quad gx = fz.$$

If a sextic scroll has three double conics belonging to an axial pencil, the common chord cannot be a double generator.

When, instead of each point of intersection being a simple self-corresponding point, one of them is a double element, the surface can have no double generators, nor can the surface have a rectilinear directrix. A third conic cannot lie in any plane passing through the points of intersection of the first two. Through the point (0, 1, 0, 0) pass four generators.

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* "Recherches sur les surfaces réglées." Paris, 1867. See p. 156.

If the conics have but one point of intersection, their equations may be written

$$x = \mu, \quad y = \mu^2, \quad z = 0, \quad w = 1,$$
$$x = 0, \quad y = \lambda^2, \quad z = \lambda, \quad w = 1 - \varkappa\lambda^2,$$

and the equation of the correspondence is

$$a\lambda^2\mu^2 + b\lambda^2\mu + c\lambda\mu^2 + d\lambda^2 + e\lambda\mu + f\mu^2 = 0.$$

The residual curve is of order 6 and cuts each generator twice. Type LX. The surface belongs to a complex if $d + f = 0$ and the complex becomes special, if in addition, $e = 0$. The axis of the complex is defined by

$$y = 0, \quad bx - cz + f = 0.$$

The residual curve is of order 5 and cuts each generator once. Type LXI.

When $\varkappa = 0$ and a correspondence of the form

$$a\lambda^2\mu + b\lambda^2 + c\lambda\mu^2 + d\lambda\mu + e\mu^3 + f\mu^2 = 0$$

exists, the conic in $x = 0$ is a triple conic and the other is a double conic. The residual curve is also a conic. It has one point in common with each of the given conics, hence the surface belongs to the unicursal (2, 2) family with one double element. When $d = 0$ and $f + b = 0$, the third conic lies in the common tangent plane of the two given conics at a point of intersection, and touches the double conic. Type LXII.

§7.—*Dual of* (3, 3) *Types.*

Scrolls of the dual of the (3, 3) type may be generated by joining corresponding points of two twisted cubics or unicursal plane cubics by a (1, 1) correspondence, making three varieties. In case the scroll contains two plane cubics, it counts as a separate type, LXIII. No curve of order less than three can be a simple directrix of the scroll.

A double generator may exist, type LXIV, a triple generator, type LXV. The scrolls may also be generated by joining corresponding points of a cubic and a unicursal quartic which intersects it in a self-corresponding point, and similarly for curves of order 5 or 6.

One cubic curve may be replaced by a triple line, but this form has already been considered; every scroll containing a triple line must also contain a cubic curve or another triple line. When the line meets the cubic, it becomes a four-

fold line, and the residual curve is of order 4. It is a distinct type, LXVI. A double generator may be present, type LXVII, or two; the residual curve becoming a double conic, LXVIII.

§8.—*Sextic Developables.*

The forms of sextic developables are classified in the doctor dissertation of Professor Schwarz* according to characteristics of cuspidal edge and double curve. The five possible types are discussed in Salmon's Geometry, pp. 314–318. The general case is a specialization of LX and its dual; two forms appear from the latter when one or two inflexional tangents are present. They are most easily studied by the methods employed in types XXII to XXIX. The developables of order 6 are all unicursal (planar). They will not be included in the following table.

§9.—*Table of Forms of Unicursal Sextic Scrolls.*

By writing $\varkappa c_n^m$ for $\varkappa$ distinct curves of order n, each counting as an m-fold curve on the scroll, g^l as an l-fold generator, any line-symbol with a bar over it for coincident tangent planes and $[c_3]$ for a plane cubic, the characteristics of the scrolls obtained may be expressed as follows:

I.	c_1^5,	XIV.	$c_1^4 + c_4^2$,
II.	$(c_1^4 + g^2)$,	XV.	$c_1^4 + c_3^2 + g^2$,
III.	$(c_1^3 + 2g^2)$,	XVI.	$c_1^4 + c_2^2 + 2g^2$,
IV.	$(c_1^2 + g^3)$,	XVII.	$c_1^4 + c_1^2 + 3g^2$,
V.	$(c_1^1 + g^4)$,	XVIII.	$c_1^4 \equiv c_1'^2 + 3g^2$,
VI.	$(c_1^3 + 2\bar{g}^2)$,	XIX.	$c_1^2 + c_{9,4}^2$,
VII.	$(c_1^3 + g' + g_1^2)$,	XX.	$c_1^2 + c_{8,3}^2 + g^2$,
VIII.	$(c_1^2 + \bar{g}^3)$,	XXI.	$c_1^2 + c_{7,2}^2 + 2g^2$,
IX.	$(c_1^1 + g + \bar{g}_1^3)$,	XXII.	$c_{10,3}^2$,
X.	$(c_1^1 + \bar{g}^2 + \dot{g}_1^2)$,	XXIII.	$c_1^3 + c_7^2$,
XI.	$(c_1^1 + \bar{g}^4)$,	XXIV.	$c_1^3 + c_6^2 + g^2$,
XII.	$(c_1^1 + c_1^5)$,	XXV.	$c_1^3 + c_5^2 + 2g^2$,
XIII.	$c_{10,4}^2$,	XXVI.	$3c_1^3 + 4g^2$,

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* "De superficiebus in planum explicabilibus primorum septem ordinum." Crelle, Vol. 64.

XXVII. $2c_1^3 + g^3 + g'^2$,

XXVIII. $c_1^3 \equiv c_1'^3 + 4g^2$,

XXIX. $c_1^3 \equiv c_1'^2 + g^3 + g'^2$,

XXX. $c_{10}^2 + c_1^1$,

XXXI. $(c_1^1 + g) + c_9^2$.

XXXII. $(c_1^1 + g) + c_6^2 + c_3^2$,

XXXIII. $(c_1^1 + g) + c_6^2 + [c_3^2]$,

XXXIV. $(c_1^1 + g) + 3c^2$,

XXXV. $(c_1^1 + g^2) + c_7^2$,

XXXVI. $(\overline{c_1^1 + g^2)} + c_6^2$,

XXXVII. $(c_1^1 + g^2) + 2c_3^2$,

XXXVIII. $(c_1^1 + g^2) + c_3^3$,

XXXIX. $(c_1^1 + g^3) + c_4^3$,

XL. $\overline{(c_1^1 + g^3}) + c_3^2$,

XLI. $(c_1^1 + g^3) + [c_3^2]$,

XLII. $c_{10}^2 + c_2^1$,

XLIII. $c_9^2 + c_2^1 + g^2$,

XLIV. $c_8^2 + c_2^1 + 2g^2$,

XLV. $(c_1^2 + g) + c_7^2$,

XLVI. $(c_1^2 + g) + c_6^2 + g^2$,

XLVII. $(c_1^2 + g) + c_5^2 + 2g^2$,

XLVIII. $(c_1^2 + g^2) + c_4^2$,

XLIX. $(c_1^2 + g^2) + c_3^2 + g^2$,

L. $(c_1^2 + g^2) + c_2^2 + 2g^2$,

LI. $c_8^2 + c_2^2$,

LII. $c_7^2 + c_2^2 + g^2$,

LIII. $c_6^2 + c_2^2 + 2g^2$,

LIV. $c_1^4 + c_2^2 + \overline{2g^2}$,

LV. $c_1^2 + c_2^2 + c_7^2$,

LVI. $c_1^2 + c_2^2 + c_6^2 + g^2$,

LVII. $2c_2^2 + c_5^2 + g^2$,

LVIII. $3c_2^2 + c_3^2 + g^2$,

LIX. $2c_2^2 + c_4^2 + g^2 + c_1^2$,

LX. $2c_2^2 + c_6^2$,

LXI. $2c_2^2 + c_5^2 + c_1^2$,

LXII. $2c_2^2 + c_2^3$,

LXIII. $c_{10}^2 + [2c_3^1]$,

LXIV. $c_9^2 + 2c_3^1 + g^2$,

LXV. $c_7^2 + g^3 + [2c_3^1]$,

LXVI. $(c_1^3 + g^2) + c_4^2$,

LXVII. $(c_1^3 + g^2) + c_3^2 + g^2$,

LXVIII. $(c_1^3 + g^2) + c_2^2 + 2g^2$.

CORNELL UNIVERSITY, *Dec.*, 1901.

On the Forms of Sextic Scrolls of Genus One.

By Virgil Snyder.

1. The following theorems, which were established in my previous paper on unicursal sextic scrolls, will be made use of:

(1.) The complete nodal curve is of order 9.

(2.) Every generator cuts four others.

(3.) Every curve lying on the surface and such that a single generator passes through each point is of genus 1.

The method employed throughout will be that of algebraic correspondence between the points of two curves. If both curves are unicursal, the correspondence must be elliptic; if the curves are elliptic, the correspondence must be rational.

§1.—*General Form and* (3, 3) *Correspondence.*

2. The general sextic scroll of genus 1 can be obtained by joining corresponding points of two binodal quartic curves by means of straight lines. The quartics must have the same characteristic and must intersect in two points which are self-corresponding in the (1, 1) relation connecting the points of the two curves. Varieties exist according as two, one or no plane exists containing three generators.

The equations may be written in the form

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z}{z_1},$$

in which $$x_0 = \wp(u), \; y_0 = \wp'(u), \; x_1 = \frac{A_1 \wp(u) + B_1 \wp'(u)}{C \wp(u) + D \wp'(u)},$$

with similar expressions for y_1, z_1 having the same denominator. This is type I.

The planes of the two directrix curves may intersect in a double generator. The residual nodal curve is now of order 8; it has a triple point lying on the double generator. Type II.

3. The line of intersection may be a triple generator. The nodal curve must now break up into two triple lines since any plane section which passes through the triple generator cuts a non-singular cubic from the surface, hence it cannot meet the nodal curve except on the triple generator. Two cases exist according as the directrices are skew or coincident. The general equation of the first type is

$$z^3 f_3(x, y) + z^2 wx f_2(x, y) + zw^2 x^2 f_1(x, y) + cw^3 x^3 = 0,$$

type III, and of the second is

$$x^3 \phi_3(x, y) + x^2 \phi_2(x, y)(yz - axw) + x\phi_1(x, y)(yz - axw)^2 + b(yz - axw)^3 = 0,$$

type IV. The line $x = 0$, $z = 0$ is the triple generator in each case.

These two surfaces can be most easily generated as follows: Consider a line $x = 0$, $y = 0$ and a non-singular cubic curve $c_3 = 0$ lying in the plane $z = 0$. Let a pencil of lines in $z = 0$, whose vertex is not on c_3, be made projective with the points of the line. Any line of the pencil will cut c_3 in three points. Connect each of these points with the point on the directrix which corresponds to the given line of the pencil. The connecting lines will generate a sextic scroll having the line x, y for triple directrix and contained in a linear congruence. When the vertex of the pencil is at the point in which the directrix cuts the plane of the cubic, the congruence becomes special. These surfaces contain an infinite number of non-singular cubic curves, all of which can be cut from the same cone.

4. A more general correspondence may be established as follows: Let a series of conics be passed through four fixed basis points, three of which lie on the cubic. The equation of the pencil will be $c_3\mu + \kappa_2 = 0$. Any conic of the pencil will cut the cubic in three points besides the basis points. Connect each of these three points with the point $x = 0$, $y = 0$, $z = \mu$ by means of straight lines. The resulting sextic scroll will have the line x, y for triple directrix.

The residual nodal curve is of order 6; it cuts the triple line 3 times. Each generator cuts the sextic in two points.* Type V.

When two points of intersection of the cubic and a conic are collinear with the origin and the correspondence is such that this conic is associated with the origin, a double generator exists. The nodal curve is now a quintic which Salmon has called of the first kind. It intersects the triple directrix twice and the double generator three times. Type VI.

5. Finally, by making a similar correspondence between the points of a line and those of a binodal quartic curve, a scroll can be defined having two double generators. The residual nodal curve is a quartic of the second kind which cuts the triple directrix in one point. Type VII.

6. Two more types, analogous to III and IV, can be obtained by setting up an elliptic (3, 3) correspondence between the points of two straight lines which may be skew or coincident. If $\frac{y}{x} = \lambda$ and $\mu = \frac{z}{w}$, then $f(\lambda^3, \mu^3) = 0$ must have three finite double elements. If $\lambda = \frac{y}{x}$, $\mu = \frac{xz - ayw}{x}$, then $\sum_{r=0}^{3} \phi_{6-2r}(\lambda)\,\mu^r = 0$ must have three finite double elements. The former is type VIII and the latter is type IX.

§2.—*Scrolls containing a Multiple Conic.*

7. Let a conic and a line which intersects it in one point be put in (3, 2) correspondence in such a way that the point of intersection is a self-corresponding double element.

Let the equations of the line be

$$x = 0, \quad y = 0, \quad z = \mu,$$

and of the conic $$x = \lambda, \quad y = \lambda^2, \quad z = 0.$$

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* This curve is the cuspidal edge of the planar developable $at^4 + bt^3 + \ldots = 0$, in which $a, b, \ldots$ are linear functions of x, y, z. See Salmon's "Geometry," p. 296.

A generator is then defined by

$$\frac{x}{\lambda} = \frac{y}{\lambda^2} = \frac{\mu - z}{\mu},$$

from which

$$\lambda = \frac{y}{x}, \quad \mu = \frac{yz}{yw - x^2}.$$

The most general relation between λ and μ of the form defined is

$$\lambda^2 f_3(\mu) + \lambda\mu f_2(\mu) + \mu^2 f_1(\mu) = 0,$$

from which, after removing the factor y^2,

$$f_3(yz, yw - x^2) + xzf_2(yz, yw - x^2) + x^2z^2 f_1(yz, yw - x^2) = 0,$$

which is the general equation of the surface. Since the point of intersection, regarded as a point on the conic, has one non-coincident corresponding point on the line, the latter is a simple generator and double directrix. The triple line and the triple conic make up the whole nodal curve. Type X.

If the curve and the line be put in (3, 3) correspondence having the point of intersection for a self-corresponding triple point, the line will be a triple directrix but not a generator. The equation is

$$\lambda^3 f_3(\mu) + \lambda^2\mu f_2(\mu) + \lambda\mu^2 f_1(\mu) + a\mu^3 = 0,$$

or, after removing the factor y^3,

$$f_3(yz, yw - x^2) + xzf_2(yz, yw - x^2) + x^2z^2 f_1(yz, yw - x^2) + az^3x^3 = 0.$$

Type XI. These are the only forms having triple conics.

8. The next series is that in which the conic is double and the line multiple. Suppose λ, μ have the same meaning as above and the correspondence be (2, 2). The form of the equation is

$$y^2z^2\phi_2(x, y) + yz(yw - x^2)f_2(x, y) + (yw - x^2)^2\psi_2(x, y) = 0.$$

The line x, y is a double generator and a double directrix. Any plane through it will contain two other generators which intersect on the double conic. The residual nodal line is an additional double generator lying in the tangent plane through the fourfold line and consecutive to the latter. Any plane section of the surface cuts from the nodal line a fourfold point which counts as seven double points. It is a tacnode with two simple branches passing through it in different directions. Type XII.

If the correspondence be (2, 3) with a double element, the form of the equation is

$$\mu^2 f_3(\lambda) + \lambda\mu f_2(\lambda) + \lambda\phi_2(\lambda) = 0,$$

in which f_3, f_2, ϕ_2 are restricted so that a double element exists. In particular, if ϕ_2 is of the form u^2 and u is a factor of f_2, the double generator lies in the plane of the conic.

The triple directrix counts as single generator. The fourfold line, the double conic and the double generator make up the whole nodal curve. The equation is

$$yz^2 f_3(x, y) + yzf_1(x, y)\phi_1(x, y)(yw - x^2) + (yw - x^2)^2\phi_1^2(x, y) = 0. \text{ Type XIII.}$$

Let there be (2, 4) correspondence between the points of the conic and the straight line such that the point of intersection is a self-corresponding double element and having one other double element. The directrix is now fourfold and is not a generator. In particular, if the nodal generator lie in the plane of the double conic, the equation is of the form

$$z^2 f_4(x, y) + z(xw - y^2) f_2(x, y)\,.\,\phi_1(x, y) + (xw - y^2)^2\phi_1^2(x, y) = 0. \text{ Type XIV.}$$

It will be observed that the last three equations are all of the same type; the scrolls, however, are essentially different. In XII, two generators issue from each point of the directrix, and for two different points one of the generators coincides with the directrix; in XIII, three generators issue from each point and only one generator coincides with the directrix; finally, in XIV, four generators issue from each point, and none of them coincides with the directrix. These are the only types having a double conic and a multiple directrix which intersects it.

9. Consider a (2, 2) correspondence between the points of a conic and of a line which does not intersect it. Let the equations of the conic be

$$x = \lambda, \quad y = \frac{1}{\lambda}, \quad z = 0$$

and of the line $$z = 0, \quad y = 0, \quad z = \mu.$$

The equations of the line joining λ to μ give

$$\lambda^2 = \frac{y}{x}, \quad \mu = \frac{\lambda z}{\lambda - x}.$$

Substitute these values in a general (2, 2) correspondence between λ, μ. The result may be written in the form

$$(f_2(x, z, w)\,x + \phi_2(x, z, w)\,y + axy^2)^2 = xy\,(\psi_2(x, y, w) + yf_1(x, z, w))^2,$$

from which the residual curve of order 6 is directly evident. The sextic cuts the double directrix twice and the conic twice. Each generator cuts the sextic twice. Type XV.

When the values of λ which correspond to $\mu = 0$ give points on the conic which are collinear with the origin the line joining these points is a double generator. The residual quintic curve cuts the double directrix but once. The expression for the (2, 2) correspondence is now of the form

$$\phi_2(\lambda)\,\mu^2 + f_2(\lambda)\,\mu + c\,(\lambda^2 - \varkappa^2) = 0,$$

from which the equations of the surface and of the nodal quintic can be at once obtained. Type XVI.

10. Two conics lying in different planes but having two points of intersection, generate a sextic scroll when put in (2, 2) correspondence with each point of intersection as a single self-corresponding point.

Let the equations of the conics be

$$x = \varkappa\mu, \quad y = \mu^2, \quad z = 0,$$
$$x = 0, \quad y = \lambda^2, \quad z = \lambda.$$

The equations of the line joining the point λ to the point μ are

$$\lambda x = \mu\lambda - \mu z, \quad \varkappa y - \varkappa\lambda z = \mu x.$$

The parameters λ, μ are connected by the relation

$$a\mu^2\lambda + b\mu\lambda^2 + c\lambda^2 + d\mu^2 + e\mu\lambda + f\lambda + g\mu = 0.$$

By writing

$$bx - a\varkappa z \equiv l_1, \quad \varkappa^2 y\,(cy + fz) \equiv yu, \quad \varkappa z^2 - x^2 - \varkappa y \equiv f_2,$$
$$a\varkappa^2 zy - 2\varkappa bxy + cx^2 + d\varkappa^2 z^2 - e\varkappa xz \equiv \phi_2,$$
$$b\varkappa^2 y^2 - 2\varkappa cxy + e\varkappa^2 yz - f\varkappa xz + g\varkappa^2 z^2 \equiv \psi_2,$$

the equation of the scroll may be expressed by the vanishing of a determinant,

$$\begin{vmatrix} l_1 & \phi_2 & \psi_2 & yu & 0 \\ 0 & xl_1 & \phi_2 & \psi_2 & u \\ w & f_2 & \varkappa xy & 0 & 0 \\ 0 & xw & f_2 & \varkappa xy & 0 \\ 0 & 0 & xw & f_2 & \varkappa x \end{vmatrix} = 0,$$

which still contains the extraneous factor z^3. The residual curve is of order 5 and is cut by every generator twice. A multiple generator cannot appear. Type XVII.

The double quintic may break up into a quartic and a double directrix line. For, express the condition that any generator should cut a given line. The resultant is a (2, 2) correspondence between λ, μ of the kind here treated, but with restricted coefficients. The quartic curve cuts the double directrix in two points and every generator cuts it in one point. The scroll may be defined as generated by all the common secants of the line and the two conics. The quartic curve cannot further degrade, for, if a double generator appeared, the scroll would be unicursal; if a second directrix line were present, the scroll would belong to a linear congruence and consequently have no nodal curve. Type XVIII.

The scroll may have a third double conic, which may be determined as follows: Suppose $Ax + By + Cz = 0$ is the plane of the new conic, which latter will also pass through the origin. Solve for the point in which a variable generator cuts this plane,

$$x = \frac{-\lambda\mu\,(\varkappa C + \varkappa B\lambda)}{\mu\,(\mu B + \varkappa A) - \lambda\,(\varkappa C + \varkappa B\lambda)},\ y = \ldots.,\ z = \ldots.$$

Now, consider any quadric surface, with undetermined coefficients, which passes through the origin; impose the condition that the point just found also lies on this surface.

There are condition equations to solve for the unknowns linearly, and putting the values found for the coefficients in the relation between λ, μ, it becomes a (2, 2) correspondence of the kind here needed and not having a double element, hence the new conic is double.

The residual nodal curve is a cubic which is cut once by every generator. It cannot further degrade. If a double directrix line exists, the possibility of a third nodal conic is excluded. The surface may be generated by the common secants of three conics, all of which pass through one point and each pair having one further point common. Type XIX.

10. A sextic scroll exists having a double conic and a residual curve of order 7, which is cut in three points by each generator. Take a binodal quartic in $z = 0$.

$$f(x, y, w) = (ax^2 + bxw + cw^2)\, y^2 + (a'x^2 + b'xw + c'w^2)\, yw + (a''x^2 + b''xw + c''w^2)\, w^2 = 0,$$

and a pencil of conics $\phi_2 + \varkappa\psi_2 = 0$ passing through the two nodes and two other fixed points of the quartic.

Make the conics of the pencil projective with the points of the conic $x = 0$, $yz = w^2$ such that the nodes, considered as points on the new conic correspond to conics of the pencil which touch one of the branches of the quartic at the respective node. The lines joining corresponding points will generate a scroll of the type desired. Type XX.

§3.—*General* (2, 4) *Correspondence.*

11. In No. 10, if the conic $x = 0$, $yz = w^2$ be replaced by the range

$$x = 0, \quad y = 0, \quad z = \varkappa w,$$

the line will be a double directrix. The residual curve is of order 8 which cuts the double directrix twice and each generator three times. The equation can be written directly by eliminating x_1, y, $\varkappa$, w_1 between the equations

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{\varkappa w - z}{\varkappa}, \quad f(x_1, y_1, w_1) = 0, \quad \phi_2(x_1, y_1, w_1) + \varkappa\psi_2(x_1, y_1, w_1) = 0,$$

and dividing out six extraneous linear factors $x^2 = 0$, $y^2 = 0$ and the first members of the equations of the planes containing the line x, y and one or the other of the basis points of the pencil of conics. Type XXI.

If $c'' = 0$, the quartic intersects the multiple directrix which now becomes a triple line composed of a double directrix and single generator. The residual curve is a sextic which cuts the triple line three times and every other generator twice. Type XXII.

Suppose now that the line be

$$x = 0, \quad w = 0, \quad z = \varkappa y.$$

Then it becomes a double directrix and double generator. The residual curve is a twisted cubic which cuts the multiple directrix twice but every other generator once. The scroll may be generated by lines joining points of a (2, 2) correspondence between a twisted cubic and one of its double secants, the points of intersection both being simple self-corresponding elements. Type XXIII.

Now, let one of the basis points of the pencil of conics be taken off the quartic. The line x, y will become a triple directrix. Let the conic which is tangent to one of the branches at the node correspond to the same point on the directrix. The latter is triple directrix and simple generator. The residual nodal curve is a twisted cubic which cuts the multiple line twice and every other generator once. Type XXIV. This form can be generated by means of a (2, 3) correspondence between a twisted cubic and a double secant, one point of intersection being doubly self-corresponding, the other simply.

Now, suppose the quartic to have a cusp at x, w. Let the pencil of conics have two basis points not on the quartic, and let the conic which touches the cuspidal tangent at the cusp correspond to the same point on the directrix. The latter becomes a fourfold directrix and the residual curve is a twisted cubic cutting the directrix and each generator once. The surface may be generated by means of a (2, 4) correspondence between a twisted cubic and one of its double secants, the points of intersection both being self-corresponding double points. Type XXV.

An illustration is furnished by the curve

$$x = \lambda(\lambda - 1), \quad y = \lambda^2(\lambda - 1), \quad z = \lambda$$

and the line

$$x = 0, \quad y = 0, \quad z = \mu.$$

From the equations of the line joining the point μ to the point λ one obtains

$$\lambda = \frac{y}{x}, \quad \mu = \frac{y\,(x^2 + z(y - x))}{x^3 + yw\,(y - x)}.$$

The (2, 2) correspondence between λ, μ must be satisfied by (0, 0) and by (1, 1); similar restrictions exist for the other forms.

12. If, in forms XXI to XXV, the correspondence were established by means of a pencil of lines instead of conics, a double generator would exist. Let the quartic be defined as in No. 10 and the points on the lines x, y be projectively associated with the lines

$$z = 0, \quad x + \mu w = 0,$$

wherein

$$\kappa = \frac{\alpha\mu + \beta}{\gamma\mu + \delta}.$$

The elimination of x gives

$$ay^2u^4 + (by^2 + a'xy)\mu^3 + (cy^2 + b'xy + a''x^2)\mu + (c''xy + b''x^2)\mu + c''x^2 = 0,$$
$$(\gamma z - aw)\mu^2 + (\delta z + ax - \beta w)\mu + \beta x = 0.$$

The factor x^2 can be removed from the μ eliminant, the other factor of which defines the surface.

The line $y = 0$, $\gamma z - aw = 0$, corresponding to the line joining the two nodes, is a double generator. The residual nodal curve is of order 7, it cuts the double directrix once and each generator three times. Type XXVI.

When $c'' = 0$, the quartic intersects the multiple directrix, which now becomes a simple generator. The residual curve is a quintic which cuts the multiple directrix twice and each generator twice. Type XXVII.

If, in the general case $(c'' \neq 0)$ $\beta = 0$, the line

$$\gamma z - aw = 0, \quad \delta z + ax = 0$$

is a fourfold directrix; the residual nodal curve is composed of the two double generators $x = 0$, $z = 0$; $y = 0$, $\gamma z - aw = 0$. This is the most general elliptic (2, 4) scroll which is contained in a linear congruence. The equation may be more easily written in the form

$$f(\lambda^4, \mu^2) = 0, \quad \lambda = \frac{y}{x}, \quad \mu = \frac{z}{w},$$

in which f has two finite double points.* Type XXVIII.

If the two directrices coincide, the equation may be written, if now $\lambda = \frac{y}{x}$, $\mu = \frac{xz - ayw}{xw}$,

$$\phi(\lambda, \mu) = \sum_{r=0}^{3} f_{6-2r}(\lambda)\mu^r = 0,$$

and ϕ has two finite double points. Type XXIX.

§4.—*Scrolls having a Plane Double Cubic.*

13. Consider a nodal cubic and a straight line passing through the node but not lying in the plane of the cubic. Let their points, which can be rationally

* See Journal, Vol. XXIII, p. 166, and Bulletin American Math. Society, Vol. 5, p. 343.

expressed in terms of parameters, be put in (2, 2) correspondence in such a way that both values of λ defining the node of the cubic correspond to the same point regarded as a point on the line.

Let $z = 0$, $axyw + f_3(x, y) = 0$ be the equations of the cubic, and $x = 0$, $y = 0$ be those of the line. Then

$$\frac{y}{x} = \lambda, \quad \frac{axyz}{axyw + f_3(x, y)} = \mu$$

are to be put in (2, 2) correspondence of the form

$$f_2(\lambda)\mu^2 + \phi_2(\lambda)\mu + \kappa\lambda = 0,$$

which gives for the equation of the surface

$$f_2(x, y)\, a^2xyz^2 + \phi_2(x, y)\, az\,(axyw + f_3(x, y)) + x\,(axyw + f_3(x, y))^2 = 0,$$

The line x, y is a double directrix and a double generator, two other generators passing through each point. The fourfold line and the nodal cubic constitute the whole of the double curve. Type XXX.

Now, consider a (3, 2) correspondence between the same elements which is restricted to the form

$$\phi_3(\lambda)\mu^2 + \lambda\mu\phi_2(\lambda) + \kappa\lambda^2 = 0.$$

The equation of the surface is

$$\phi_3(x, y)\, a^2xz^2 + az\phi_2(x, y)(axyw + f_3(x, y)) + \kappa\,(axyw + f_3(x, y))^2 = 0.$$

This only differs from the preceding case in the form of ϕ_3; the previous type is derived from this one when the coefficient of x^3 is zero. The configuration of nodal lines is the same in the two cases, but the latter is an essentially different type, because three generators distinct from the line itself issue from each point of the multiple directrix. The fourfold line is now triple directrix and simple generator. Type XXXI.

If, finally, λ, μ be connected by a (4, 2) correspondence of the restricted form

$$\phi_4(\lambda)\mu^2 + \lambda\mu\phi_2(\lambda) + \kappa\lambda^2 = 0,$$

the equation of the surface becomes

$$\phi_4(x, y)\, a^2z^2 + az\phi_2(x, y)(axyw + f_3(x, y)) + \kappa\,(axyw + f_3(x, y))^2 = 0.$$

In this case four generators issue from each point of the multiple directrix which is itself not a generator. Type XXXII.

If $\mu = \dfrac{ax^2z}{ax^2w + f_3(x, y)}$, the plane double cubic has a cusp at the point in which it meets the directrix line.

§5.—*Table of Forms of Double Curves.*

I.	c_9^2.
II.	$c_8^2 + g^2$.
III.	$2d^3 + g^3$.
IV.	$d^3 \equiv d'^3 + g^3$.
V.	$c_6^2 + d^3$.
VI.	$c_5^2 + d^3 + g^2$.
VII.	$c_4^2 + d^3 + 2g^2$.
VIII.	$2d^3 + 3g^2$.
IX.	$d^3 \equiv d'^3 + 3g^2$.
X.	$c_2^3 + (d^2, g)$.
XI.	$c_2^3 + d^3$.
XII.	$c_2^2 + g^2 + (d^2, g^2)$ (tacnodal).
XIII.	$c_2^2 + g^2 + (d^3, g)$.
XIV.	$c_2^2 + g^2 + d^4$.
XV.	$c_2^2 + c_6^2 + d^2$.
XVI.	$c_2^2 + c_5^2 + g^2 + d^2$.
XVII.	$2c_2^2 + c_5^2$.
XVIII.	$d^2 + 2c_2^2 + c_4^2$.
XIX.	$3c_2^2 + c_3^2$ (skew).
XX.	$c_2^2 + c_7^2$.
XXI.	$d^2 + c_8^2$.
XXII.	$c_6^2 + (d^2, g)$.
XXIII.	c_3^2 (skew) $+ (d^2, g^2)$.
XXIV.	c_3^2 (skew) $+ (d^3, g)$.
XXV.	c_3^2 (skew) $+ d^4$.
XXVI.	$c_7^2 + d^2 + g^2$.
XXVII.	$c_5^2 + g^2 + (d^2, g)$.
XXVIII.	$d^4 + d^2 + 2g^2$.
XXIX.	$d^4 \equiv (d^2, g^2) + 2g^2$.
XXX.	$(d^2, g^2) + c_3^2$ (plane).
XXXI.	$(d^3, g) + c_3^2$ (plane).
XXXII.	$d^4 + c_3^2$ (plane).

CORNELL UNIVERSITY, *June*, 1902.

Note on Symmetric Functions.

By E. D. Roe, Jr.

In this paper new proofs and more definite formulations of two previous theorems are given. The following notation is introduced:

$$D^{(m)}_{m+n-1-i}\,a^* = |\,i_m i_{m-1} \ldots . i_1| = \begin{vmatrix} \alpha_1^{i_m} \alpha_1^{i_{m-1}} \ldots . \alpha_1^{i_1} \\ \alpha_2^{i_m} \alpha_2^{i_{m-1}} \ldots . \alpha_2^{i_1} \\ \ldots\ldots\ldots\ldots \\ \alpha_m^{i_m} \alpha_m^{i_{m-1}} \ldots . \alpha_m^{i_1} \end{vmatrix} \tag{1}$$

$$\Delta^{(m)}_x\,a\dagger = \{x_1 x_2 \ldots . x_m\} = \begin{vmatrix} a_{x_1} & a_{x_2} & \ldots . a_{x_m} \\ a_{x_1-1} & a_{x_2-1} & \ldots . a_{x_m-1} \\ \ldots & \ldots & \ldots \\ a_{x_1-m+1} & a_{x_2-m+1} & \ldots . a_{x\;\,-m+1} \end{vmatrix}, \tag{2}$$

$$\left\{\begin{matrix} p_1 p_2 \ldots . p_m \\ x_1 x_2 \ldots . x_m \end{matrix}\right\} = \text{the coefficient of } a_{p_1} a_{p_2} \ldots . a_{p_m} \text{ in } \{x_1 x_2 \ldots . x_m\} \tag{3}$$

$$= \text{“} \quad \text{“} \quad \text{of } b_{p_1} b_{p_2} \ldots . b_{p_m} \text{ in } \Delta^{(m)}_x b.$$

§1.—*Theorem* I.

The product of a symmetric function $\Sigma\alpha_1^{p_1}\alpha_2^{p_2} \ldots . \alpha_m^{p_m}$ by the alternant $|0, 1, 2, \ldots . m-1|$ is obtained by adding the p's in all possible permutations to the exponents of the columns of the alternant written as a determinant in which each line contains the powers of a single letter, thus giving the product in the general case as the sum of m ! alternants.

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* These subscript indices are abbreviations; written in full, they would contain all the i's and all the x's respectively. For restrictions upon these and other notations, see §7. † Ibid.

§2.—*Proofs of Theorem* I.

1. Muir has covered this theorem in proving a similar proposition for the product of the alternant $|q_1 q_2 \ldots q_m|$ and the preceding symmetric function from considerations of symmetry, and the fact that the product as a whole must be an alternating function.*

2. The writer has also given a proof, based on substitutions, in which it is shown that the substitutions,

$$\begin{pmatrix} 1\;2\;3 \ldots m \\ i_1\, i_2\, i_3 \ldots i_m \end{pmatrix}\begin{pmatrix} 1\;2\;3 \ldots m \\ j_1 j_2 j_3 \ldots j_m \end{pmatrix} \text{ and } \begin{pmatrix} j_1 j_2 j_3 \ldots j_m \\ i_1\, i_2\, i_3 \ldots i_m \end{pmatrix}\begin{pmatrix} j_1 j_2 j_3 \ldots j_m \\ 1\;2\;3 \ldots m \end{pmatrix},$$

when applied to the straightforward multiplication of the alternant and symmetric function, and that demanded by the theorem respectively, give the same term, and hence give the required $(m!)^2$ terms identically the same.† This proof would apply without change if the alternant $|q_1 q_2 \ldots q_m|$ had been used.

3. Professor W. H. Metzler has suggested the following proof: If we multiply the alternant $|q_1 q_2 \ldots q_m|$ by $s_p = \alpha_1^p + \alpha_2^p + \ldots \alpha_m^p$, we get, from a well-known theorem in determinants,

$$\begin{aligned}(\alpha_1^{p_1} + \alpha_2^{p_2} + \ldots \alpha_m^{p_m}) \times |q_1 q_2 \ldots q_m| \\ = |p+q_1, q_2, \ldots q_m| + |q_1, p+q_2, \ldots q_m| + \ldots + |q_1, q_2, \ldots p+q_m|. \quad (4)\end{aligned}$$

As every symmetric function of the roots can be expressed as a function of $s_1, s_2, \ldots s_m$, the symmetric function $\Sigma\alpha_1^{p_1}\alpha_2^{p_2} \ldots \alpha_m^{p_m}$, by which we are to multiply our alternant, can be expressed as

$$\phi(s_1, s_2, \ldots s_m) \equiv s_{p_1} s_{p_2} \ldots s_{p_m} + A\Sigma s_{p_1} s_{p_2} \ldots s_{p_{m-1}+p_m} + \ldots \quad (5)$$

Now, from the known properties of the coefficients in ϕ,‡ it is easily seen that in the product of our alternant by ϕ, every alternant of the form

* Muir, "Determinants." 1882, p. 176, §129.

† American Mathematical Monthly, Vol. 6 (1899), p. 25. The author there attributed theorems I and II to Professor Gordan. Professor Metzler has kindly called the writer's attention to the reference to Muir, from which it appears that Muir has the priority of publication as far at least as theorem I is concerned. It may, however, be added that in a recent letter Professor Gordan states he has used the two theorems for the last thirty years.

‡ For the exact form of the A's, see Faà di Bruno, "Binäre Formen," p. 8, or Am. Math. Monthly, Vol. 5 (1898), p. 164, or Vol. 7 (1900), p. 66.

$|q_1 + x_1, q_2 + x_2, \ldots\ldots q_m + x_m|$, where $x_1, x_2, \ldots\ldots x_m$ are not some permutation of the p's, will have zero as coefficient.

§3.—*Theorem* II.

From theorem I another may be obtained by eliminating the α's in the right member, interpreted as the roots of

$$f(x) = a_0 x^m + a_1 x^{m-1} + \ldots\ldots + a_m = 0, \tag{6}$$

by means of the theorem of corresponding matrices,* which expresses the symmetric function $a_0^n \Sigma \alpha_1^{p_1} \alpha_2^{p_2} \ldots\ldots \alpha_m^{p_m}$, as a sum of determinants of the n^{th} order in the a's.† This we shall call theorem II.‡ The developments of the next section show, however, *that theorem II can be proved independently of theorem I, and that theorem I can be made to depend upon theorem II.*

§4.—*More Exact Formulation of Theorems* I *and* II.

In order to throw out the numerous permutations which occur in the preceding proofs, i. e. *in order to deal with combinations only instead of permutations,* and to define the coefficient of any one alternant in the α's or determinant in the a's, according as theorem I or theorem II is desired, the following method may be used:

If
$$\phi(x) = b_0 x^n + b_1 x^{n-1} + \ldots\ldots + b_n, \tag{7}$$
then of the matrices

$$\left|\begin{array}{cccccc} a_0 & a_1 & \ldots\ldots a_m & 0 & \ldots\ldots & 0 \\ 0 & a_0 & \ldots\ldots & a_m & 0 \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & a_0 & & \ldots\ldots & a_m \end{array}\right| = A, \ n \text{ lines}, \tag{8}$$

$$\left|\begin{array}{cccccc} b_0 & b_1 & \ldots\ldots b_n & 0 & \ldots\ldots & 0 \\ 0 & b_0 & \ldots\ldots & b_n & 0 \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots\ldots & 0\, b_0 & & \ldots\ldots & b_n \end{array}\right| = B, \ m \text{ lines}, \tag{9}$$

$$\left|\begin{array}{cccc} \alpha_1^{m+n-1} & \alpha_1^{m+n-2} & \ldots\ldots & 1 \\ \alpha_2^{m+n-1} & \alpha_2^{m+n-2} & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_m^{m+n-1} & \alpha_m^{m+n-2} & \ldots\ldots & 1 \end{array}\right| = C, \ m \text{ lines}, \tag{10}$$

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* Gordan, "Invariantentheorie," Vol. 1, p. 95.

† Am. Math. Monthly, Vol. 6 (1899), p. 2.

‡ In the writer's former paper (l. c.) both theorems were treated practically as one.

A and C are corresponding matrices, and $\begin{vmatrix} A \\ B \end{vmatrix}$ is the Sylvestrian determinant resultant of f and ϕ. By Laplace's development,

$$R_{f,\phi} = \begin{vmatrix} A \\ B \end{vmatrix} = \Sigma(-1)_\mu \Delta_x^{(n)} a \Delta_i^{(m)} b.^* \tag{11}$$

By the theorem of corresponding matrices,

$$\Delta_x^{(n)} a = (-1)^\mu \lambda D_i^{(m)} a, \tag{12}$$

whence†

$$R_{f,\phi} = \lambda \Sigma D_i^{(m)} a \Delta_i^{(m)} b = \lambda \Sigma D_{m+n-1-i}^{(m)} a \Delta_{m+n-1-i}^{(m)} b. \tag{13}$$

In each determinant $\Delta_{m+n-1-i}^{(m)} b$ we may pick out the term containing $b_{n-p_1} b_{n-p_2} \ldots . b_{n-p}$, and we may thus rearrange our sum with reference to terms of this form and write

$$R_{f,\phi} = \lambda \Sigma b_{n-p_1} b_{n-p_2} \ldots . b_{n-p_m}$$
$$\times \Sigma \begin{Bmatrix} n-p_1 \; n-p_2 \ldots . \; n-p_m \\ m+n-1-i_m \ldots . \; m+n-1-i_1 \end{Bmatrix} |i_m \, i_{m-1} \ldots . \, i_1|. \tag{14}$$

Again,

$$R_{f,\phi} = a_0^n (b_0 \alpha_1^n + b_1 \alpha_1^{n-1} + \ldots b_n)(b_0 \alpha_2^n + b_1 \alpha_2^{n-1} + \ldots b_n) \ldots (b_0 \alpha_m^n + b_1 \alpha_m^{n-1} + \ldots b_n)$$
$$= a_0^n \Sigma b_{n-p_1} b_{n-p_2} \ldots . b_{n-p_m} \Sigma \alpha_1^{p_1} \alpha_2^{p_2} \ldots . \alpha_m^{p_m}. \tag{15}$$

By equating the coefficients of $b_{n-p_1} b_{n-p_2} \ldots . b_{n-p_m}$ in both developments of the resultant, we have

$$a_0^n \Sigma \alpha_1^{p_1} \alpha_2^{p_2} \ldots . \alpha_m^{p_m}$$
$$= \lambda \Sigma \begin{Bmatrix} n-p_1 \; n-p_2 \ldots . \; n-p_m \\ m+n-1-i_m \ldots . \; m+n-1-i_1 \end{Bmatrix} |i_m \, i_{m-1} \ldots . \, i_1|. \tag{16}$$

By corresponding matrices,

$$a_0^n = \lambda |m-1, \; m-2, \ldots . \; 1, \; 0|; \tag{17}$$

also, we have

$$|m-1, \; m-2, \ldots . \; 1, \; 0| = (-1)^{\frac{m(m-1)}{2}} |0, \; 1, \; 2, \ldots . \; m-1|; \tag{18}$$

---

* The subscript complexes of indices x and i together make up all the indices $0, 1, 2, \ldots m+n-1$, i. e. these determinants contain no pair of corresponding columns from the two matrices. Also any a, with an index less than 0 or greater than m, is zero; similarly for the b's. See §7.

† Compare Gordan, "Invariantentheorie," Vol. 1, p. 184.

similarly with $|i_m i_{m-1} \dots i_1|$; and we have in succession

$$\begin{Bmatrix} n-p_1 & \dots & n-p_m \\ m+n-1-i_m & \dots & m+n-1-i_1 \end{Bmatrix} = (-1)^{\frac{m(m-1)}{2}} \begin{Bmatrix} p_1 p_2 & \dots & p_m \\ i_m i_{m-1} & \dots & i_1 \end{Bmatrix}$$

$$= (-1)^{\frac{m(m-1)}{2}+\frac{m(m-1)}{2}} \begin{Bmatrix} p_1 p_2 & \dots & p_m \\ i_1\ i_2 & \dots & i_m \end{Bmatrix} = \begin{Bmatrix} p_1 p_2 & \dots & p_m \\ i_1\ i_2 & \dots & i_m \end{Bmatrix} \tag{19}$$

by taking first the complements of the indices with respect to n and then double transpositions of the elements of the determinant

$$\begin{vmatrix} b_{i\ -m+1} & \dots & b_{i_1-m+1} \\ b_{i_m-m+2} & \dots & b_{i_1-m+2} \\ \dots & \dots & \dots \\ b_{i_m} & \dots & b_{i_1} \end{vmatrix}.$$

Using in (16) the values obtained in (17), (18) and (19), we have, as the expression of theorem I,

$$1.\ |0, 1, 2, \dots m-1|\Sigma\alpha_1^{p_1}\alpha_2^{p_2} \dots \alpha_m^{p_m} = \Sigma \begin{Bmatrix} p_1 p_2 & \dots & p_m \\ i_1\ i_2 & \dots & i_m \end{Bmatrix} |i_1 i_2 \dots i_m|. \tag{20}$$

By using the theorem of corresponding matrices in (16), *or by directly expanding* (11) *after the manner of* (14), we obtain the expression for theorem II,

$$2.\ \ \alpha_0^n \Sigma\alpha_1^{p_1}\alpha_2^{p_2} \dots \alpha_m^{p_m} = \Sigma(-1)^{\mu} \begin{Bmatrix} p_1 p_2 & \dots & p_m \\ i_1\ i_2 & \dots & i_m \end{Bmatrix} \{i\ _{+1} i_{m+2} \dots i_{m+n}\}, \tag{21}$$

which expresses a symmetric function of the α's homogeneously as a sum of determinants of the a's of the n^{th} order.

§5.—*The Coefficients* $\begin{pmatrix} q_1 q_2 & \dots & q_n \\ 0^n p_1 p_2 & \dots & p_m \end{pmatrix}$* *in Terms of the Coefficients* $\begin{Bmatrix} p_1 p_2 & \dots & p_m \\ i_1\ i_2 & \dots & i_m \end{Bmatrix}$.

If we expand the right member of (21) and collect the coefficient of the term $a_{q_1} a_{q_2} \dots a_{q_n}$, which we denote by $\begin{pmatrix} q_1 q_2 & \dots & q_n \\ 0^n p_1 p_2 & \dots & p_m \end{pmatrix}$ according to notation previously used elsewhere,† we have, since we shall show in §7 that μ is

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* The author's dissertation, "Die Entwickelung der Sylvester'schen Determinante nach Normal-Formen." Leipzig, B. G. Teubner, 1898, pp. 4 and 39, and Am. Math. Monthly, Vol. 6 (1899), pp. 55, 57, 104 et seq.

† Ibid.

constant and equal to $p_1 + p_2 + \dots p_m$,

$$\begin{pmatrix} q_1 q_2 \dots q_n \\ 0^n p_1 p_2 \dots p_m \end{pmatrix}$$

$$= (-1)^{p_1+p_2+\dots p_m} \Sigma \begin{Bmatrix} p_1 p_2 \dots p_m \\ i_1 \; i_2 \dots i_m \end{Bmatrix} \begin{Bmatrix} q_1 q_2 & \dots q_n \\ i_{m+1} i_{m+2} & \dots i_{m+n} \end{Bmatrix}. \quad (22)$$

Hence (21) becomes

$$a_0^n \Sigma a_1^{p_1} a_2^{p_2} \dots a_m^{p_m} = \Sigma \begin{pmatrix} q_1 q_2 \dots q_n \\ 0^n p_1 p_2 \dots p_m \end{pmatrix} a_{q_1} a_{q_2} \dots a_{q_n}$$

$$= (-1)^{p_1+p_2+\dots p_m} \Sigma\Sigma \begin{Bmatrix} p_1 p_2 \dots p_m \\ i_1 \; i_2 \dots i_m \end{Bmatrix} \begin{Bmatrix} q_1 q_2 \dots q_n \\ i_{m+1} \dots i_{m+n} \end{Bmatrix} a_{q_1} a_{q_2} \dots a_{q_n}. \quad (23)$$

§6.—*The Coefficients* $q_1 q_2 \dots q_n | n - p_1 \, n - p_2 \dots n - p_m$* *in Terms of the Coefficients* $\begin{Bmatrix} p_1 p_2 \dots p_m \\ i_1 \; i_2 \dots i_m \end{Bmatrix}$.

If we substitute the value of the symmetric function, as given by (21), in the development of the resultant as expressed by (15), and collect the coefficient of $a_{q_1} a_{q_2} \dots a_{q_n} b_{n-p_1} b_{n-p_2} \dots b_{n-p_m}$, which has also been previously denoted elsewhere† by $q_1 q_1 \dots q_n | n - p_1 \, n - p_2 \dots n - p_m$, we have

$$q_1 q_2 \dots q_n | n - p_1 \, n - p_2 \dots n - p_n$$

$$= (-1)^{p_1+p_2+\dots p_m} \Sigma \begin{Bmatrix} p_1 p_2 \dots p_m \\ i_1 \; i_2 \dots i_m \end{Bmatrix} \begin{Bmatrix} q_1 q_2 & \dots q_n \\ i_{m+1} i_{m+2} & \dots i_{m+n} \end{Bmatrix}, \quad (24)$$

and, by using this value, (15) becomes

$$R_{f,\phi} = (-1)^{p_1+p_2+\dots p_m} \Sigma\Sigma\Sigma \begin{Bmatrix} p_1 p_2 \dots p_m \\ i_1 \; i_2 \dots i_m \end{Bmatrix} \begin{Bmatrix} q_1 q_2 \dots q_n \\ i_{m+1} \dots i_{m+n} \end{Bmatrix}$$

$$\times a_{q_1} a_{q_2} \dots a_{q_n} b_{n-p_1} b_{n-p_2} \dots b_{n-p_m}. \quad (25)$$

§7.—*Restrictive Relations.*

The foregoing summations are restricted by the following conditions on the indices and exponents. For (20) and (21):‡

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* The author's dissertation, "Die Entwickelung der Sylvester'schen Determinante nach Normal-Formen." Leipzig. B. G. Teubner, 1898, pp. 4 and 39, and Am. Math. Monthly, Vol. 6 (1899), pp. 55, 57, 104 et seq.

† Ibid.

‡ For (22) and (23) $q_1 + q_2 + \dots q_n = p_1 + p_2 + \dots p_m$, and for (23) $mn \geqq p_1 + p_2 + \dots p_m \geqq 0$.

$$n \geqq p_1 \geqq p_2 \geqq \ldots\ldots \geqq p_m \geqq 0, \tag{26}$$

$$p_1 + m - 1 \geqq i_m > i_{m-1} > \ldots\ldots > i_1 \geqq 0, \tag{27}$$

$$i_{m+n} > i_{m+n-1} > \ldots\ldots > i_{m+1}, \tag{28}$$

$$i_1 + i_2 + \ldots\ldots i_m = \frac{m(m-1)}{2} + p_1 + p_2 + \ldots\ldots p_m. \tag{29}$$

$i_{m+1}, i_{m+2}, \ldots\ldots i_{m+n}$ are the indices of the elements of the first line of the determinant corresponding to $|i_m i_{m-1} \ldots\ldots i_1|$. The distinction of a and b may be dropped in (20) and (21) and then with respect to

$$\begin{Bmatrix} p_1 p_2 \ldots\ldots p_m \\ i_1\ i_2\ \ldots\ldots\ i_m \end{Bmatrix}, \ a_{n+i} = a_{-i} = 0;$$

with respect to

$$\{i_{m+1} i_{m+2} \ldots\ldots i_{m+n}\}, \ a_{m+i} = a_{-i} = 0. \tag{30}$$

Since the sum of the indices $0, 1, 2, \ldots\ldots m + n - 1$, each increased by 1 is $\frac{(m+n)(m+n+1)}{2}$, we have $m + n - i_m + m + n - i_{m-1} + \ldots\ldots m + n - i_1 + i_{m+1} + 1 + \ldots\ldots + i_{m+n} + 1 = \frac{(m+n)(m+n+1)}{2}$, or

$$i_{m+1} + \ldots\ldots i_{m+n} - \frac{n(n-1)}{2} = i_1 + i_2 + \ldots\ldots i_m - \frac{m(m-1)}{2}$$
$$= (\text{by } (29))\, p_1 + p_2 + \ldots\ldots p_m. \tag{31}$$

Now

$$\mu = (i_{m+1} + 1) - 1 + (i_{m+2} + 1) - 2 \ldots\ldots (i_{m+n} + 1) - n$$
$$= i_{m+1} + i_{m+2} + \ldots\ldots i_{m+n} - \frac{n(n-1)}{2}, \tag{32}$$

or by (31) and (32),

$$\mu = p_1 + p_2 + \ldots\ldots p_m. \tag{33}$$

§8.—*The Calculation of the Coefficient*

$$\begin{Bmatrix} p_1 p_2 \ldots\ldots p_m \\ i_1\ i_2\ \ldots\ldots\ i_m \end{Bmatrix}.$$

A recurrence formula for the calculation of $\begin{Bmatrix} p_1 p_2 \dots p_m \\ i_1 i_2 \dots i_m \end{Bmatrix}$ is already involved in the nature of the coefficient as expressed. If r numbers be common to the two series $p_1, p_2, \dots p_m$ and $i_1, i_2, \dots i_m$, so that

$$\begin{aligned} i_{\lambda_1} &= p_{\lambda_1}, \\ i_{\lambda_2} &= p_{\lambda_2}, \\ &\dots\dots \\ i_{\lambda_r} &= p_{\lambda_r}, \end{aligned} \tag{34}$$

we have, by expanding the determinant $\{i_1 i_2 \dots i_m\}$ in terms of the elements of the first line,

$$\begin{Bmatrix} p_1 p_2 \dots p_m \\ i_1 i_2 \dots i_m \end{Bmatrix}$$
$$= \sum_{\kappa=1}^{\kappa=r} (-1)^{\lambda_\kappa - 1} \begin{Bmatrix} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_{\kappa-1}} p_{\lambda_{\kappa+1}} \dots p_{\lambda_m} \\ i_1 - 1, i_2 - 1, \dots i_{\lambda_{\kappa-1}} - 1, i_{\lambda_{\kappa+1}} - 1 \dots i_m - 1 \end{Bmatrix}, \tag{35}$$

a coefficient of order m, expressed as a sum of several of order $m-1$. If no numbers of the two series are common, the coefficient is zero. If a lower index becomes negative, the coefficient is also zero. The order of the upper indices is indifferent for calculation. It is obvious that $\begin{Bmatrix} x \\ x \end{Bmatrix} = 1$.

§9.—*Examples.*

1. $\begin{Bmatrix} 012 \\ 123 \end{Bmatrix} = \begin{Bmatrix} 02 \\ 12 \end{Bmatrix} - \begin{Bmatrix} 01 \\ 02 \end{Bmatrix} = - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = -2.$

2. $\begin{Bmatrix} 012 \\ 024 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 13 \end{Bmatrix} - \begin{Bmatrix} 01 \\ -13 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2 \end{Bmatrix} = 1.$

3. $\begin{Bmatrix} 0^2 13 \\ 0127 \end{Bmatrix} = \begin{Bmatrix} 013 \\ 016 \end{Bmatrix} - \begin{Bmatrix} 0^2 3 \\ -116 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 05 \end{Bmatrix} - \begin{Bmatrix} 03 \\ -15 \end{Bmatrix} = 0.$

4. $\begin{Bmatrix} 0123 \\ 0345 \end{Bmatrix} = \begin{Bmatrix} 123 \\ 234 \end{Bmatrix} - \begin{Bmatrix} 012 \\ -134 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 23 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 13 \end{Bmatrix} = - \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 2 \\ 2 \end{Bmatrix} = -2.$

5. $\begin{Bmatrix} 0123 \\ 0156 \end{Bmatrix} = 0.$

Since, by (31) and (33), the equation

$$i_{m+1} + i_{m+2} + \dots \, i_{m+n} - \frac{n(n-1)}{2} = \mu = p_1 + p_2 + \dots \, p_m$$

exists, or

$$i_{m+1} + i_{m+2} - 1 + i_{m+3} - 2 + \dots \, i_{m+n} - (n-1) = \mu, \qquad (36)$$

i. e., since the sum of the principal diagonal indices of the determinant $\{i_{m+1} i_{m+2} \dots i_{m+n}\}$ is equal to the weight of the symmetric function, a fact which we also know otherwise from general theory, it is best in practice to form the different sets of principal diagonal indices of the determinants $\{i_{m+1} i_{m+2} \dots i_{m+n}\}$ first, as these are the sets most easily formed; by the addition of 0, 1, 2, ... $n-1$ respectively to these, we get the first line indices; we take next the complements of these with respect to $m+n-1$, and lastly, the remaining indices of the set 0, 1, 2 .... $m+n-1$, not found among the complements for the indices of the alternants, and lastly, write these indices under the indices of the symmetric function for the coefficient of the alternant and determinant when μ is even, but the negative of this coefficient for the coefficient of the determinant when μ is odd. It is also necessary to take $m=\mu$ in order to get a general result, while n is taken as the order of the symmetric function.

6. It is required to find $|01234|\Sigma\alpha_1^3\alpha_2^2$ and $a_0^3\Sigma\alpha_1^3\alpha_2^2$. We have the following calculation:

Principal Diagonal Indices. Sum $=\mu=5$.	First Line Indices. Sum $=\mu+\frac{n(n-1)}{2}=8$.	Complements with respect to $m+n-1$. Sum $=mn+\frac{n(n-1)}{2}-\mu=13$.	Remaining Indices. Sum $=\frac{m(m-1)}{2}+\mu=15$.	Coefficients.
$\lambda_1, \lambda_2, \lambda_3$	$\lambda_1, \lambda_2+1, \lambda_3+2$	$\lambda-(m+n-1)$		
005	017	067	12345	$\begin{Bmatrix} 0^3 23 \\ 12345 \end{Bmatrix}$
014	026	157	02346	$\begin{Bmatrix} 0^3 23 \\ 02346 \end{Bmatrix}$
113	125	256	01347	$\begin{Bmatrix} 0^3 23 \\ 01347 \end{Bmatrix}$
122	134	346	01257	$\begin{Bmatrix} 0^3 23 \\ 01257 \end{Bmatrix}$
023	035	247	01356	$\begin{Bmatrix} 0^3 23 \\ 01356 \end{Bmatrix}$

The calculation of the coefficients follows:

$$\begin{Bmatrix} 0^3 23 \\ 12345 \end{Bmatrix} = -\begin{Bmatrix} 0^3 3 \\ 0234 \end{Bmatrix} + \begin{Bmatrix} 0^3 2 \\ 0134 \end{Bmatrix} = -\begin{Bmatrix} 0^2 3 \\ 123 \end{Bmatrix} + \begin{Bmatrix} 0^2 2 \\ 023 \end{Bmatrix}$$

$$= -\begin{Bmatrix} 0^2 \\ 01 \end{Bmatrix} + \begin{Bmatrix} 02 \\ 12 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = -2.$$

$$\begin{Bmatrix} 0^3 23 \\ 02346 \end{Bmatrix} = \begin{Bmatrix} 0^2 23 \\ 1235 \end{Bmatrix} = -\begin{Bmatrix} 0^2 3 \\ 024 \end{Bmatrix} + \begin{Bmatrix} 0^2 2 \\ 014 \end{Bmatrix} = -\begin{Bmatrix} 03 \\ 13 \end{Bmatrix} + \begin{Bmatrix} 02 \\ 03 \end{Bmatrix} = +\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 2 \\ 2 \end{Bmatrix} = 2.$$

$$\begin{Bmatrix} 0^3 23 \\ 01347 \end{Bmatrix} = \begin{Bmatrix} 0^2 23 \\ 0236 \end{Bmatrix} = \begin{Bmatrix} 023 \\ 125 \end{Bmatrix} = -\begin{Bmatrix} 03 \\ 04 \end{Bmatrix} = -\begin{Bmatrix} 3 \\ 3 \end{Bmatrix} = -1.$$

$$\begin{Bmatrix} 0^3 23 \\ 01257 \end{Bmatrix} = \begin{Bmatrix} 0^2 23 \\ 0146 \end{Bmatrix} = \begin{Bmatrix} 023 \\ 035 \end{Bmatrix} = \begin{Bmatrix} 23 \\ 24 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 3 \end{Bmatrix} = 1.$$

$$\begin{Bmatrix} 0^3 23 \\ 01356 \end{Bmatrix} = \begin{Bmatrix} 0^2 23 \\ 0245 \end{Bmatrix} = \begin{Bmatrix} 023 \\ 134 \end{Bmatrix} = -\begin{Bmatrix} 02 \\ 03 \end{Bmatrix} = -1.$$

By substituting the values of the preceding coefficients, we have, as the results of the calculation, and as illustrations of theorems I and II (Formulas (20) and (21)),

$$|01234|\Sigma\alpha_1^3\alpha_2^2 = -2|12345| + 2|02346| - |01347| + |01257| - |01356|, \qquad \text{I.}$$

$$a_0^3\Sigma\alpha_1^3\alpha_2^2 = 2\begin{vmatrix} a_0 & a_1 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & a_5 \end{vmatrix} - 2\begin{vmatrix} a_0 & a_2 & 0 \\ 0 & a_1 & a_5 \\ 0 & a_0 & a_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_5 \\ a_0 & a_1 & a_4 \\ 0 & a_0 & a_3 \end{vmatrix}$$

$$- \begin{vmatrix} a_1 & a_3 & a_4 \\ a_0 & a_2 & a_3 \\ 0 & a_1 & a_2 \end{vmatrix} + \begin{vmatrix} a_0 & a_3 & a_5 \\ 0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}. \qquad \text{II.}$$

SYRACUSE UNIVERSITY, *October* 25, 1901.

*The Double-six Configuration Connected with the Cubic Surface, and a Related Group of Cremona Transformations.**

BY EDWARD KASNER.

The double-six configuration, consisting of two sets of six lines each so related that any line of either set intersects all except a corresponding line of the other set, first presented itself to Schlaefli † in the study of the twenty-seven lines upon the cubic surface. In this paper the configuration is investigated in itself, i. e., independently of the cubic surface determined by it, the latter being introduced only incidentally in the final section. The starting point is the theory of five collinear lines. Denoting these by L_1, L_2, L_3, L_4, L_5 and their common tractor by M_0, then each quadruple as L_2, L_3, L_4, L_5 has (in addition to M_0) a proper tractor as M_1; thus, five new lines M_1, M_2, M_3, M_4, M_5 are obtained. That these derived lines are themselves collinear, having a common tractor L_0, was observed incidentally by Schlaefli‡ and verified by Cayley || The proof given in §4 is direct and simple. The twelve lines L_i, M_i, form a double-six.

The relations between the anharmonic ratios of the thirty points P_{ik} and

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* Read in different form before the American Mathematical Society, February 23, 1901.

† "An attempt to determine the twenty-seven lines upon a surface of third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface" (Quarterly Journal of Mathematics, Vol. II, 1858, pp. 55–65, 110–120). ‡ L. c., p. 117.

|| "On the Double-Sixers of a Cubic Surface" (Collected Papers, Vol. VII, pp. 316–330; Quarterly Journal of Mathematics, Vol. X, 1870, pp. 58–71). Cf. also "On Dr. Wiener's Model of a Cubic Surface with 27 Real Lines; and on the Construction of a Double-Sixer" (Collected Papers, Vol. VIII, pp. 366–384; Cambridge Philosophical Transactions, Vol. XII, Part I, 1873, pp. 366–383).

thirty planes Π_{ik},* determined by the twelve lines, are discussed in §6; all the ratios are expressible rationally in terms of a fundamental set of four (§7). The Cremona group discussed in §8 arises from the transformations which are induced in the fundamental set by permutations of the lines. In §§9, 10, certain results concerning the double-six and the general cubic surface due to Schur† and Reye‡ are presented from a more simple point of view by employing the relations between the anharmonic ratios.

§1.—*The Coordinate System.* The quintuple§ of collinear lines contains 19 arbitrary constants; but, by properly choosing the system of coordinates, these may be reduced to four projectively essential constants. For this purpose take, as the fundamental tetrahedron of the system, that determined by the points P_{25}, P_{15}, P_{20}, P_{10}, so that the coordinates of these points are

$$1,\ 0,\ 0,\ 0\ ;\ 0,\ 1,\ 0,\ 0\ ;\ 0,\ 0,\ 1,\ 0\ ;\ 0,\ 0,\ 0,\ 1$$

respectively. The unit point is still at our disposal and may be chosen so as to satisfy any three conditions. Since P_{35} is collinear with P_{15} and P_{25}, its coordinates are of the form x_1, x_2, 0, 0; and similarly, those of P_{30} are of the form 0, 0, x_3, x_4. If, then, we take the coordinates of these points to be 1, 1, 0, 0 and 0, 0, 1, 1 respectively, we, in effect, impose only two conditions upon the unit point. Consider, finally, the plane determined by the line L_1 and by the point of intersection of the line L_5 and the plane Π_{25}; its coordinates are of the form u_1, 0, u_3, 0, so that if we take these to be 1, 0, —1, 0 we impose a third condition upon the unit point. The system of coordinates is now completely determined.

§2.—*The Five L Lines.* The four constants which are necessary for the representation of the quintuple may be introduced as follows: The points

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* P_{ik} is the point of intersection of the lines L_i, M_k; Π_{ik} is the plane of the same line.

† "Ueber die durch collineare Grundgebilde erzeugten Curven und Flächen" (Mathematische Annalen, Vol. 18, 1881, pp. 1–32).

‡ "Beziehung der allgemeinen Fläche dritter Ordnung zu einer covarianten Fläche dritter Classe" (Mathematische Annalen, Vol. 55, 1901, pp. 257–264).

§ It is assumed throughout the paper that of the five lines constituting the collinear quintuple, no two intersect, and no four have a double tractor, so that the hyperboloid determined by any three does not touch either of the remaining lines.

P_{40}, P_{50}, P_{45} lie on the sides of the fundamental tetrahedron, so that each is representable in terms of a single parameter; the coordinates are of the form $0, 0, 1, l$; $0, 0, 1, m$; $1, \lambda, 0, 0$ respectively. The point of intersection of L_5 and Π_{25} lies in the planes $x_1 - x_3 = 0$ and $x_4 = 0$, so that its coordinates are of the form $1, \mu, 1, 0$.

The four constants l, m, λ, μ may be interpreted very simply as anharmonic ratios. The coordinates of the five points of the line M_0 and of the five planes through M_0 are found to be

$$\left.\begin{array}{lccccclcccc}
P_{10}: & 0 & 0 & 0 & 1; & \qquad & \Pi_{10}: & 1 & 0 & 0 & 0; \\
P_{20}: & 0 & 0 & 1 & 0; & & \Pi_{20}: & 0 & 1 & 0 & 0; \\
P_{30}: & 0 & 0 & 1 & 1; & & \Pi_{30}: & -1 & 1 & 0 & 0; \\
P_{40}: & 0 & 0 & 1 & l; & & \Pi_{40}: & -\lambda & 1 & 0 & 0; \\
P_{50}: & 0 & 0 & 1 & m; & & \Pi_{50}: & -\mu & 1 & 0 & 0;
\end{array}\right\} \tag{1}$$

from which follow the required interpretations:

$$\left.\begin{array}{ll}
l = (P_{10}, P_{20}, P_{30}, P_{40}), & \lambda = (\Pi_{10}, \Pi_{20}, \Pi_{30}, \Pi_{40}), \\
m = (P_{10}, P_{20}, P_{30}, P_{50}), & \mu = (\Pi_{10}, \Pi_{20}, \Pi_{30}, \Pi_{50}).
\end{array}\right\} \tag{1'}$$

The six homogeneous coordinates of each of the lines L_1, L_2, L_3, L_4, L_5 may now be calculated, since upon each we have two known points. Thus the line L_4 passes through the points P_{45} and P_{40}; its coordinates are then the minors of the array

$$\begin{vmatrix} 1 & \lambda & 0 & 0 \\ 0 & 0 & 1 & l \end{vmatrix},$$

or
$$p_{12} : p_{13} : p_{14} : p_{34} : p_{42} : p_{23} = 0 : 1 : l : 0 : -l\lambda : \lambda.$$

The table of coordinates is as follows:

$$\left.\begin{array}{lcccccc}
L_1: & 0 & 0 & 0 & 0 & 1 & 0 \\
L_2: & 0 & 1 & 0 & 0 & 0 & 0 \\
L_3: & 0 & 1 & 1 & 0 & -1 & 1 \\
L_4: & 0 & 1 & l & 0 & -l\lambda & \lambda \\
L_5: & 0 & 1 & m & m & -m\mu & \mu
\end{array}\right\} \tag{2}$$

The coordinates of the common tractor M_0 are

$$M_0: \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0. \tag{3}$$

§3.—*The Five M Lines.* To exemplify the method, consider the line M_4, which is the tractor proper to the four lines L_1, L_2, L_3, L_5. Equating the simultaneous invariants of the intersecting lines to 0, we obtain the conditions:

$$\begin{aligned}
(M_4L_1) &= p_{13} = 0,\\
(M_4L_2) &= p_{42} = 0,\\
(M_4L_3) &= p_{42} + p_{23} - p_{13} + p_{14} = 0,\\
(M_4L_5) &= p_{42} + mp_{23} - m\mu p_{13} + \mu p_{14} = 0,
\end{aligned}$$

which, with the quadratic identity

$$(M_4M_4) = p_{12}\,p_{34} + p_{13}\,p_{42} + p_{14}\,p_{23} = 0,$$

give

$$p_{12} : p_{13} : p_{14} : p_{34} : p_{42} : p_{23} = (m-\mu)^2 : 0 : m\,(m-\mu) : m^2 : 0 : -\,m\,(m-\mu).$$

The coordinates of the remaining lines are found in a similar manner.

$$\left.\begin{array}{llllllll}
M_1: & A & m\,(\lambda - l) & \lambda m\,(\lambda - 1) & \dfrac{l\lambda m^2\,(\lambda-1)(l-1)}{A} & 0 & -\,lm\,(l-1) \\[2ex]
M_2: & B & 0 & m\,(1-l) & \dfrac{m^2\,(\lambda-1)(l-1)}{B} & m\,(l-\lambda) & m\,(\lambda-1) \\[2ex]
M_3: & C & 0 & lm & \dfrac{l\lambda m^2}{C} & 0 & -\,\lambda m \\[2ex]
M_4: & (m-\mu)^2 & 0 & m\,(m-\mu) & m^2 & 0 & -\,m\,(m-\mu) \\
M_5: & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right\}\quad(4)$$

The quantities A, B, C are functions of l, m, λ, μ:

$$\left.\begin{aligned}
A &= m\,(1-l)(\mu-\lambda) - \mu\,(1-\lambda)(m-l),\\
B &= (1-\lambda)(m-l) - (1-l)(\mu-\lambda),\\
C &= \lambda m - l\mu.
\end{aligned}\right\}\qquad(5)$$

§4.—*Schlaefli's Theorem.* Five lines in general determine a linear complex. Let the equation of the linear complex to which M_1, M_2, M_3, M_4, M_5 belong be

$$D_{12}\,p_{12} + D_{13}\,p_{13} + D_{14}\,p_{14} + D_{34}\,p_{34} + D_{42}\,p_{42} + D_{23}\,p_{23} = 0,$$

then the coefficients $D_{i\kappa}$ are proportional to the determinants of the fifth order formed from the matrix whose rows are the coordinates of five lines $M_1, \ldots\ldots, M_5$.

After some reduction, these are found to be

$$D_{12} = 0, \qquad D_{13} = l\lambda\, m^2 \mu\, B^2, \qquad D_{14} = \lambda m^2 AB,$$
$$D_{34} = (m - \mu)\, ABC, \qquad D_{42} = -\, mA^2, \qquad D_{23} = lm\mu\, AB.$$

Substituting these values, we find

$$D_{12} D_{34} + D_{13} D_{42} + D_{14} D_{23} = 0,$$

i. e., the invariant of the complex vanishes. From a known theorem, then, the complex is special, and the lines belonging to it are all collinear.

The lines M_1, M_2, M_3, M_4, M_5 obtained from five collinear lines L_1, L_2, L_3, L_4, L_5 by taking the tractor proper to each set of four of the latter lines are also collinear.

The coordinates of the common tractor L_0 of the M lines are

$$L_0\colon\ (m - \mu)\, ABC, \quad -mA^2, \quad lm\mu\, AB, \quad 0, \quad l\lambda m^2 \mu\, B, \quad \lambda m^2 AB. \tag{6}$$

§5.—*The Double-six Configuration L, M, P, Π.* From the original set of five collinear lines L_1, L_2, L_3, L_4, L_5, there is thus derived a configuration of twelve lines L_0, L_1, L_2, L_3, L_4, L_5, M_0, M_1, M_2, M_3, M_4, M_5. Each L line intersects all the M lines except the one whose index is the same, and similarly, each M line intersects five of the L lines. The various pairs of intersecting lines determine 30 points and 30 planes

$$P_{i\kappa},\ \Pi_{i\kappa}, \qquad (i,\ \kappa = 0,\ 1,\ 2,\ 3,\ 4,\ 5\,;\ i \neq \kappa)$$

where $P_{i\kappa}$ denotes the point of intersection of L_i and M_κ, and $\Pi_{i\kappa}$ denotes the plane of the same two lines.

In the following, indicate by ι_0, ι_1, ι_2, ι_3, ι_4, ι_5 any permutation of the six indices 0, 1, 2, 3, 4, 5. Through each point $P_{\iota_0 \iota_1}$ there pass 9 planes

$$\Pi_{\iota_0 \iota_1}, \Pi_{\iota_0 \iota_2}, \Pi_{\iota_2 \iota_1}, \Pi_{\iota_0 \iota_3}, \Pi_{\iota_3 \iota_1}, \Pi_{\iota_0 \iota_4}, \Pi_{\iota_4 \iota_1}, \Pi_{\iota_0 \iota_5}, \Pi_{\iota_5 \iota_1};$$

and similarly, in each plane there lie 9 points. On each L line L_{ι_0} there are five points $P_{\iota_0 \iota_1}$, $P_{\iota_0 \iota_2}$, $P_{\iota_0 \iota_3}$, $P_{\iota_0 \iota_4}$, $P_{\iota_0 \iota_5}$, and through it there pass five planes $\Pi_{\iota_0 \iota_1}$, $\Pi_{\iota_0 \iota_2}$, $\Pi_{\iota_0 \iota_3}$, $\Pi_{\iota_0 \iota_4}$, $\Pi_{\iota_0 \iota_5}$; similarly with respect to the M lines. The 30 points lie by fives in 6 lines in two distinct ways, and the thirty planes pass by fives through 6 lines in two ways. Any five L lines, as well as any five M lines, are

collinear; the common tractor of $L_{\iota_0} L_{\iota_1} L_{\iota_2} L_{\iota_3} L_{\iota_4}$ is M_{ι_5}. In a certain sense, the configuration is closed; no new lines are obtained by taking the tractors of four L or four M lines, for the two tractors of L_{ι_0}, L_{ι_1}, L_{ι_2}, L_{ι_3} are M_{ι_4} and M_{ι_5} and the two tractors of M_{ι_0}, M_{ι_1}, M_{ι_2}, M_{ι_3} are L_{ι_4} and L_{ι_5}.

The configuration was derived from L_1, L_2, L_3, L_4, L_5, but it is just as well determined by any five L lines or any five M lines. In particular, it is seen that the relation between the original quintuple $L_1, \ldots, L_5$ and the derived quintuple $M_1, \ldots, M_5$ is a reciprocal one; just as the latter lines are the tractors of the former taken in fours, so the former are the tractors of the latter.

We may state the results obtained as follows:

In connection with any five collinear lines L_1, L_2, L_3, L_4, L_5, there exists a covariant line L_0, uniquely defined by the fact that it is collinear with any four of the five, but not with all five. The relation between the six lines L_0, L_1, L_2, L_3, L_4, L_5 is a symmetrical one, i. e., the covariant line of any five is the sixth. There presents itself, then, a conjugate set of six lines $M_0, \ldots, M_5$, standing in the same symmetrical relation; these lines are the tractors of L lines taken by fives. The relation between the conjugate sets is of involutory character.

§6.—*Relations between the Anharmonic Ratios.* Consider the points in which L_1 is intersected by the lines M_0, M_2 M_3, M_4, M_5. The coordinates of the point of intersection of any two intersecting lines $p_{\iota\kappa}$, $p'_{\iota\kappa}$ are proportional to the determinants formed from the array

$$\left|\begin{array}{cccc} 0 & p_{34} & p_{42} & p_{23} \\ -p_{42} & -p_{14} & 0 & p_{12} \\ -p'_{42} & -p'_{14} & 0 & p'_{12} \end{array}\right|.$$

Thus the coordinates of the point P_{13}, the intersection of the lines L_1 and M_3, are, by (2) and (3), the determinants formed from the array

$$\left|\begin{array}{cccc} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -lm & 0 & C \end{array}\right|,$$

which reduce to 0, C, 0, lm respectively.

By this method the following table of coordinates is obtained:

$$\begin{array}{lcccc}
P_{10}: & 0 & 0 & 0 & 1\;, \\
P_{12}: & 0 & B & 0 & m(1-l), \\
P_{13}: & 0 & C & 0 & lm\;, \\
P_{14}: & 0 & m-\mu & 0 & m\;, \\
P_{15}: & 0 & 1 & 0 & 0\;.
\end{array}$$

The anharmonic ratios of these collinear points are equal to the anharmonic ratios of the corresponding parameters

$$\infty, \quad \frac{m(1-l)}{B}, \quad \frac{lm}{C}, \quad \frac{m}{m-\mu}, \quad 0. \tag{7}$$

Calculating the ratio of the last four points, we find

$$(P_{12}, P_{13}, P_{14}, P_{25}) = \frac{l(1-m)}{m(1-l)}$$

But from (1),

$$(P_{20}, P_{30}, P_{40}, P_{50}) = \frac{l(1-m)}{m(1-l)}.$$

Therefore

$$(P_{12}, P_{13}, P_{14}, P_{15}) = (P_{20}, P_{30}, P_{40}, P_{50}),$$

or employing a more abbreviated notation* for the anharmonic ratios,

$$L_1(2\;3\;4\;5) = M_0(2\;3\;4\;5).$$

From the symmetry of the configuration, we have then the result:

$$L_{i_1}(i_2 i_3 i_4 i_5) = M_{i_0}(i_2 i_3 i_3 i_5), \tag{8}$$

which may be translated as follows:

The anharmonic ratio of the points in which any four of five collinear lines cut the common tractor of all five is equal to the anharmonic ratio of the points on the fifth line, which are collinear with three of the four lines.

---

* The anharmonic ratio of the four points in which the lines M_{i_2}, M_{i_3}, M_{i_4}, M_{i_5} intersect the line L, is thus denoted by $L_{i_1}(i_2 i_3 i_4 i_5)$; while that of the four planes determined by the same lines is denoted by $\overline{L}_{i_1}(i_2 i_3 i_4 i_5)$.

Another relation may be obtained by combining the above with the well-known theorem concerning the two tractors of four lines; the four points of either tractor and the four planes of the other tractor have the same anharmonic ratio. In the present case, consider the four lines L_{i_2}, L_{i_3}, L_{i_4}, L_{i_5}; their tractors are M_{i_0}, M_{i_1}, so that

$$M_{i_0}(i_2\, i_3\, i_4\, i_5) = \overline{M}_{i_1}(i_2\, i_3\, i_4\, i_5). \tag{9}$$

From (8) we have then

$$L_{i_1}(i_2\, i_3\, i_4\, i_5) = \overline{M}_{i_1}(i_2\, i_3\, i_4\, i_5), \tag{10}$$

which may be expressed:

The anharmonic ratio of the four points on one of five collinear lines which are collinear with three of the remaining lines is equal to the anharmonic ratio of the four planes determined by these remaining lines and their common tractor.

Another relation is obtained by considering the anharmonic ratio of the points P_{10}, P_{13}, P_{14}, P_{15}, which, by means of (1), is found to be

$$L_1(0\,3\,4\,5) = \frac{l\,(m-\mu)}{m\,(l-\lambda)} = \frac{1-\mu/m}{1-\lambda/l}.$$

Introducting the interpretations of l, m, λ, μ from (1), we have

$$L_1(0\,3\,4\,5) = \frac{1-\dfrac{\overline{M}_0(1\,2\,3\,5)}{M_0(1\,2\,3\,5)}}{1-\dfrac{\overline{M}_0(1\,2\,3\,4)}{M_0(1\,2\,3\,4)}},$$

or, by permuting the indices,

$$L_{i_1}(i_2\, i_3\, i_4\, i_5) = \frac{1-\dfrac{\overline{M}_{i_2}(i_1\, i_0\, i_3\, i_5)}{M_{i_2}(i_1\, i_0\, i_3\, i_5)}}{1-\dfrac{\overline{M}_{i_2}(i_1\, i_0\, i_3\, i_4)}{M_{i_2}(i_1\, i_0\, i_3\, i_4)}}. \tag{11}$$

The complete system of relations between the anharmonic ratios of the configuration consists of the fundamental relations (8), (10) *and* (11), *together with the well-known relations* between the anharmonic ratios of five collinear points or planes.*

In the first place, by means of (8), (10), (11), it is possible to express the ratios of four points or planes of one line in terms of the ratios of the points

* See, for example, M. J. M. Hill, "The Anharmonic Ratios of the Roots of a Quintic" (Proceedings of the London Mathematical Society, Vol. XIV, p. 182).

and planes of any other line. Thus:

$$
\begin{aligned}
L_1(2345) &= M_0(2345) & &= \bar{L}_0(2345) \\
&= \bar{M}_1(2345) & &= L_1(2345) \\
&= \frac{1-\dfrac{\bar{M}_2(1035)}{M_2(1035)}}{1-\dfrac{\bar{M}_2(1034)}{M_2(1034)}} & &= \frac{1-\dfrac{L_2(1035)}{\bar{L}_2(1035)}}{1-\dfrac{L_2(1035)}{\bar{L}_2(1035)}} \\
&= \frac{1-\dfrac{\bar{M}_3(1024)}{M_3(1024)}}{1-\dfrac{\bar{M}_3(1025)}{M_3(1025)}} & &= \frac{1-\dfrac{L_3(1024)}{\bar{L}_3(1024)}}{1-\dfrac{L_3(1025)}{\bar{L}_3(1025)}} \\
&= \frac{1-\dfrac{\bar{M}_4(1053)}{M_4(1053)}}{1-\dfrac{M_4(1052)}{M_4(1052)}} & &= \frac{1-\dfrac{L_4(1053)}{\bar{L}_4(1053)}}{1-\dfrac{L_4(1052)}{\bar{L}_4(1052)}} \\
&= \frac{1-\dfrac{\bar{M}_5(1042)}{M_5(1042)}}{1-\dfrac{\bar{M}_5(1043)}{M_5(1043)}} & &= \frac{1-\dfrac{L_5(1042)}{\bar{L}_5(1042)}}{1-\dfrac{L_5(1043)}{\bar{L}_5(1043)}}
\end{aligned}
$$

From this it follows that all the ratios may be expressed in terms of those connected with any one line, say M_0. But the ratios of the five points on M_0 are rationally expressible in terms of two l and m; and similarly, the ratios of the five planes through M_0 are expressible rationally in terms of two λ and μ. The four ratios l, m, λ, μ are independent, so that there can be no relations in addition to those enumerated in the above theorem.

§7.—*The Complete System of Anharmonic Ratios.*

Of the anharmonic ratios connected with the double-six, only four are independent. For a fundamental set of independent ratios we may take

$$
\left.\begin{aligned}
l &= M_0(1234), & m &= M_0(1235), \\
\lambda &= \bar{M}_0(1234), & \mu &= \bar{M}_0(1235)\cdot
\end{aligned}\right\} \tag{12}
$$

All the ratios are expressible rationally in terms of these by functions of at most the fourth degree.

The following table* gives the anharmonic ratio of any quadruple of points situated on an L line. The ratios of any quadruple of points on an M line, or of any quadruple of planes, may also be found in the same table, in virtue of the equalities (8) and (10).

$$L_0(1\,2\,3\,4) = \lambda, \qquad L_1(0\,2\,3\,4) = \frac{(m-1)\,C}{(m-l)(m-\mu)},$$

$$L_0(1\,2\,3\,5) = \mu, \qquad L_1(0\,2\,3\,5) = \frac{(1-l)\,C}{(l-\lambda)(l-m)},$$

$$L_0(1\,2\,4\,5) = \frac{\mu}{\lambda}, \qquad L_1(0\,2\,4\,5) = \frac{(l-1)(m-\mu)}{(m-1)(l-\lambda)},$$

$$L_0(1\,3\,4\,5) = \frac{1-\mu}{1-\lambda}, \qquad L_1(0\,3\,4\,5) = \frac{l\,(m-\mu)}{m\,(l-\lambda)},$$

$$L_0(2\,3\,4\,5) = \frac{\lambda\,(1-\mu)}{\mu\,(1-\lambda)}, \qquad L_1(2\,3\,4\,5) = \frac{l\,(m-1)}{m\,(l-1)},$$

$$L_2(1\,0\,3\,4) = \frac{\lambda\,(m-l)(m-\mu)}{(m-l)\,C}, \qquad L_3(1\,2\,0\,4) = -\frac{A}{mB},$$

$$L_2(1\,0\,3\,5) = \frac{\mu\,(l-\lambda)(m-l)}{(l-1)\,C}, \qquad L_3(1\,2\,0\,5) = -\frac{A}{lB},$$

$$L_2(1\,0\,4\,5) = \frac{\mu\,(l-\lambda)(m-1)}{\lambda\,(m-\mu)(l-1)}, \qquad L_3(1\,2\,4\,5) = \frac{m}{l},$$

$$L_2(1\,3\,4\,5) = \frac{m-1}{l-1}, \qquad L_3(1\,0\,4\,5) = \frac{m\,(l-\lambda)(1-\mu)}{l\,(m-\mu)(1-\lambda)},$$

$$L_2(0\,3\,4\,5) = \frac{\lambda\,(\mu-m)}{\mu\,(\lambda-l)}, \qquad L_3(2\,0\,4\,5) = \frac{(l-\lambda)(1-\mu)}{(m-\mu)(1-\lambda)},$$

$$L_4(1\,2\,3\,0) = -\frac{\lambda m\,B}{A}, \qquad L_5(1\,2\,3\,4) = l,$$

$$L_4(1\,2\,3\,5) = m, \qquad L_5(1\,2\,3\,0) = -\frac{l\mu\,B}{A},$$

$$L_4(1\,2\,0\,5) = -\frac{A}{\lambda B}, \qquad L_5(1\,2\,4\,0) = -\frac{\mu B}{A},$$

$$L_4(1\,3\,0\,5) = \frac{A}{(\lambda-1)\,C}, \qquad L_5(1\,3\,4\,0) = \frac{(\mu-1)\,C}{A},$$

$$L_4(2\,3\,0\,5) = \frac{\lambda B}{(1-\lambda)\,C}, \qquad L_5(2\,3\,4\,0) = \frac{(1-\mu)\,C}{\mu B}.$$

* A, B, C are the functions of l, m, λ, μ defined in (5).

§8.—*The Cremona Group* G_{720}. If we denote by $i_0\, i_1\, i_2\, i_3\, i_4\, i_5$ any permutation of the indices, the anharmonic ratios which correspond to the fundamental set (12) are

$$\begin{aligned} l_i &= M_{i_0}(i_1\, i_2\, i_3\, i_4), & m_i &= M_{i_0}(i_1\, i_2\, i_3\, i_5), \\ \lambda_i &= \overline{M}_{i_0}(i_1\, i_2\, i_3\, i_4), & \mu_i &= \overline{M}_{i_0}(i_1\, i_2\, i_3\, i_5), \end{aligned}$$

But from §7 these are rational functions of $l,\ m,\ \lambda,\ \mu$:

$$\begin{aligned} l_i &= f_i(l,\ m,\ \lambda,\ \mu), & m_i &= \phi_i(l,\ m,\ \lambda,\ \mu), \\ \lambda_i &= \psi_i(l,\ m,\ \lambda,\ \mu), & \mu_i &= \chi_i(l,\ m,\ \lambda,\ \mu), \end{aligned}$$

where $f_i,\ \psi_i,\ \phi_i,\ \chi_i$ denote rational (in general fractional) functions of the fourth or lower degree. Thus to every literal substitution of the six indices

$$S_i:\quad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ i_0 & i_1 & i_2 & i_3 & i_4 & i_5 \end{pmatrix} \tag{13}$$

corresponds a rational transformation of the four fundamental invariants

$$T_i:\quad \left.\begin{aligned} l' &= l_i = f_i(l,\ m,\ \lambda,\ \mu), & m' &= m_i = \phi_i(l,\ m,\ \lambda,\ \mu), \\ \lambda' &= \lambda_i = \psi_i(l,\ m,\ \lambda,\ \mu), & \mu' &= \mu_i = \chi_i(l,\ m,\ \lambda,\ \mu). \end{aligned}\right\} \tag{14}$$

It is now to be shown that the system of transformations T so obtained form a group.

Consider any two substitutions S_i and S_j:

$$S_j:\quad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ j_0 & j_1 & j_2 & j_3 & j_4 & j_5 \end{pmatrix} = \begin{pmatrix} i_0 & i_1 & i_2 & i_3 & i_4 & i_5 \\ k_0 & k_1 & k_2 & k_3 & k_4 & k_5 \end{pmatrix},$$

their product S_k, and the corresponding transformations $T_i,\ T_j,\ T_k$. Then we have, in the first place,

$$l_k = f_k(l,\ m,\ \lambda,\ \mu), \text{ etc.},$$

and in the second place, since the permutation $k_0 \ldots k_5$ is obtained from $i_0 \ldots i_5$ by the substitution S_j,

$$l^k = f_j(l_i,\ m_i,\ \lambda_i,\ \mu_i) = f_j(f_i,\ \phi_i,\ \psi_i,\ \chi_i), \text{ etc.}$$

Therefore,

$$\begin{aligned} f_j(f_i,\ \phi_i,\ \psi_i,\ \chi_i) &= f_k, & \phi_j(f_i,\ \phi_i,\ \psi_i,\ \chi_i) &= \phi_k, \\ \chi_j(f_i,\ \phi_i,\ \psi_i,\ \chi_i) &= \chi_k, & \chi_j(f_i,\ \phi_i,\ \psi_i,\ \chi_i) &= \chi_k, \end{aligned}$$

which expresses the essential group property of the system of transformations T,

$$T_i\, T_j = T_k.$$

In particular, it follows that the inverse of T_i is also a transformation of the set, and therefore the transformations are birational or Cremona transformations.

The transformations T, induced by the permutations of the indices, constitute a group G_{720} of Cremona transformations which is holoedrically isomorphic with the symmetric substitution group on six letters.

A set of generators may be obtained by writing down the transformations which correspond to generators of the symmetric group $S\,(0\ 1\ 2\ 3\ 4\ 5)$. Thus the following five form a convenient set of generators:

$$\begin{array}{rcccc}
(34) \sim T_1: & \dfrac{1}{l} & \dfrac{m}{l} & \dfrac{1}{\lambda} & \dfrac{\mu}{\lambda} \\
(23)(45) \sim T_2: & 1-m & 1-l & 1-\mu & 1-\lambda \\
(45) \sim T_3: & m & l & \mu & \lambda \\
(12) \sim T_4: & \dfrac{1}{l} & \dfrac{1}{m} & \dfrac{1}{\lambda} & \dfrac{1}{\mu} \\
(01) \sim T_5: & \dfrac{(\mu-1)(\lambda m - l\mu)}{(\mu-\lambda)(m-\mu)} & \dfrac{(\lambda-1)(\lambda m - l\mu)}{(\mu-\lambda)(l-\lambda)} & \dfrac{(m-1)(\lambda m - l\mu)}{(m-l)(m-\mu)} & \dfrac{(l-1)(\lambda m - l\mu)}{(m-l)(l-\lambda)}
\end{array}$$

The subgroup G_{120}, consisting of the transformations corresponding to those permutations which leave the index 0 unaltered, is essentially a group in two variables, since the generators T_1, T_2, T_3, T_4 transform l, m and λ, μ cogrediently. The group in two variables so obtained is identical with the cross-ratio group discussed by E. H. Moore* and H. E. Slaught.† The total group G_{720}, however, cannot be expressed in two variables. Expressed in homogeneous form, five variables are necessary,

$$t_1 : t_2 : t_3 : t_4 : t_5 = \lambda : \mu : l : m : 1.$$

* "The Cross-Ratio Group of $n!$ Cremona Transformations of Order $n-3$ in Flat Space of $n-3$ Dimensions" (American Journal of Mathematics, Vol. XXII, 1900, p. 279).

† "The Cross-Ratio Group of 120 Quadratic Cremona Transformations of the Plane" (American Journal of Mathematics, Vol. XXII, 1900, pp. 343–380). The generators K, L, M, T', given on p. 344, correspond to the generators T_1, T_2, T_3, T_4 employed above.

The generators then become

$$
\begin{array}{lllll}
T_1: & t_3t_5 & t_2t_3 & t_1t_5 & t_1t_4 & t_1t_3 \\
T_2: & t_5-t_2 & t_5-t_1 & t_5-t_4 & t_5-t_2 & t_5 \\
T_3: & t_2 & t_1 & t_2 & t_3 & t_5 \\
T_4: & t_2t_3t_4t_5 & t_1t_3t_4t_5 & t_1t_2t_4t_5 & t_1t_2t_3t_5 & t_1t_2t_3t_4
\end{array}
$$

$$
T_5: \left\{ \begin{array}{ll}
(t_2-t_1)(t_3-t_1)(t_4-t_5)(t_1t_4-t_2t_3) & (t_2-t_1)(t_4-t_2)(t_3-t_5)(t_1t_4-t_2t_3) \\
\quad (t_4-t_3)(t_3-t_1)(t_2-t_5)(t_1t_4-t_2t_3) & (t_4-t_3)(t_4-t_3)(t_1-t_5)(t_1t_4-t_2t_3) \\
\quad t_5\,(t_2-t_1)(t_4-t_3)(t_3-t_1)\,(t_4-t_2). &
\end{array} \right.
$$

§9.—*The Automorphic Correlation* Ω.* Consider two quintuples of collinear lines L_1, L_2, L_3, L_4, L_5 and $L'_1, L'_2, L'_3, L'_4, L'_5$. Denote the common tractor of the first set by M_0, the point of intersection of M_0 and L_i by P_i, the plane of the same pair of lines by Π_i, with a similar notation for the second set. We now prove the following

LEMMA. *The two quintuples are homographic when*

$$(P_1\,P_2\,P_3\,P_4\,P_5) \overset{\circ}{\wedge} (P'_1\,P'_2\,P'_3\,P'_4\,P'_5),$$
$$(\Pi_1\,\Pi_2\,\Pi_3\,\Pi_4\,\Pi_5) \overset{\circ}{\wedge} (\Pi'_1\,\Pi'_2\,\Pi'_3\,\Pi'_4\,\Pi'_5);$$

they are correlative when

$$(P_1\,P_2\,P_3\,P_4\,P_5) \overset{\circ}{\wedge} (\Pi'_1\,\Pi'_2\,\Pi'_3\,\Pi'_4\,\Pi'_5),$$
$$(\Pi_1\,\Pi_2\,\Pi_3\,\Pi_4\,\Pi_5) \overset{\circ}{\wedge} (P'_1\,P'_2\,P'_3\,P'_4\,P'_5).$$

It will be sufficient to give the proof of the second part, since that of the first follows the same scheme. Assuming the second set of relations, it is to be shown that there is a correlation transforming the first quintuple into the second. The general space correlation, or reciprocity, contains 15 parameters. The number of correlations transforming M_0 into M'_0 is therefore ∞^{11}. If we impose the

* Schur, l. c., p. 12.

conditions that Π'_1, Π'_2, Π'_3 shall correspond to P_1, P_2, P_3, and that Π'_1, Π'_2, Π'_3 shall correspond to P_1, P_2, P_3, the number of correlations is reduced to ∞^5. Each of those correlations, in virtue of the assumed relations, will then transform P_4, P_5, Π_4, Π_5 into Π'_4, Π'_5, P'_4, P'_5, and, therefore, any line of the first quintuple L_i is transformed into a line belonging to the pencil determined by P'_i and Π'_i. If, then, we require the correlation to transform $L_1, \ldots, L_5$ into $L'_1, \ldots, L'_5$, we impose just five additional (linear) conditions and the correlation is completely determined. Similarly, it may be proved that when the first set of relations holds, there is a definite homography transforming the first quintuple into the second.

We apply the lemma to the quintuples $M_1, \ldots, M_5$ and $L_1, \ldots, L_5$ with the common tractors L_0 and M_0 respectively. From the relations (10) it follows that the conditions of the second part of the lemma are fulfilled, so that there exists a correlation Ω transforming $M_1 \ldots M_5$ into $L_1 \ldots L_5$.

The quintuple of lines $M_1, \ldots, M_5$, obtained from a quintuple of collinear lines $L_1, \ldots, L_5$ by taking the tractor proper to each set of four of the latter lines, is correlative to the original quintuple.

This is an extension of Schlaefli's theorem, the latter merely states that $M_1, \ldots, M_5$ are collinear, while the above includes in addition the relations (10).

Consider now the effect of the correlation Ω upon the double-six. The common tractor L_0 of the one quintuple becomes the common tractor M_0 of the other. Taking the tractor proper to each set of four, it follows that $L_1, \ldots, L_5$ transform into $M_1, \ldots, M_5$ and, therefore, M_0 into L_0. Hence P_{ik} becomes Π_{ki} and Π_{ik} becomes P_{ki}.

There exists for the double-six configuration an automorphic correlation Ω which interchanges the lines L_i, M_i and the points and planes P_{ik}, Π_{ki}.

The character of Ω is determined as follows: Since Ω transforms P_{ik} into Π_{ki} and Π_{ki} into P_{ik}, the collineation Ω^2 leaves the points P_{ik} invariant; in particular, the five points P_{10}, P_{20}, P_{13}, P_{23}, P_{45}, no four of which are coplanar, are invariant, and, therefore, Ω^2 is the identical transformation, i. e., Ω is involutorial. There are, however, two species of involutorial transformations in space of three dimensions, namely: polarities, in which corresponding points and planes are conjugate with respect to a proper quadric surface; and null-systems, in

which all corresponding points and planes are incident. The latter species is at once excluded in the present case by the fact that the plane Π_{ki} corresponding to the point P_{ik} does not pass through it. Therefore,

The automorphic correlation Ω of the double-six is a polarity, i e., there exists a proper quadric surface Q with respect to which the points P_{ik} and the plane Π_{ki} are conjugate.

§10 —*The Cubic Surfaces F and Φ.** The number of arbitrary constants involved in a set of five collinear lines is 19, which is equal to the number of constants in a quaternary cubic form. In fact, it is easy to show that through five collinear lines $L_1 \ldots . L_5$ there pass a single surface F of the order, and also a single surface Φ of the third class. Thus a surface of the third order is determined by the 19 points, P_{10}, P_{12}, P_{13}, P_{14}, P_{20}, P_{21}, P_{23}, P_{24}, P_{30}, P_{31}, P_{32}, P_{34}, P_{40}, P_{41}, P_{42}, P_{43}, P_{51}, P_{52}, P_{53}; but the first four points are on M_0, therefore, the entire line M_0 lies in the surface, and in particular the point P_{50}; then each of the lines $L_1 \ldots . L_5$ has four points in the surface, and so lies in it. The surfaces F and Φ pass through the 12 lines of the double-six, as may be proved by showing that each line has four points in common with F and four planes in common with Φ.

It is well known† that the 27 lines on F consist of the 12 lines L_i, M_i and the 15 lines $c_{ik} = c_{ki}$ obtained as the intersections of the pairs of planes Π_{ik}, Π_{ki}. From the principle of duality, it follows that the 27 lines on Φ consist of the 12 lines L_i, M_i and the 15 lines $d_{ik} = d_{ki}$ obtained by joining the pairs of points P_{ik}, P_{ki}.

The relation between F and Φ is obtained by considering the polarity Ω defined in the previous section: Ω transforms F into a surface of the third class passing through the double-six; but there is only one such surface, namely, Φ; therefore, Ω transforms F into Φ, and similarly, Φ is transformed into F.

Through five collinear lines there pass a single surface F of the third order, and a single surface Φ of the third class. These surfaces intersect in the double-six determined by the five lines, and are reciprocally related by the polarity Ω.‡

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* Reye, l. c.

† Schlaefli, l. c.

‡ Compare the corresponding theorem with regard to three arbitrary lines: Through three lines there pass a single surface of the second order and a single surface of the second class; these two surfaces coincide.

The number of double-six's which can be formed from the 27 lines on a cubic surface is 36.* The preceding theorem thus determines, for the general cubic surface F of third order, a set of 36 irrational quadric covariants $[Q]$, and a corresponding set of 36 cubic contravariants $[\Phi]$.

COLUMBIA UNIVERSITY, NEW YORK, *November*, 1901.

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* Schlaefli, l. c., p. 115.

Untersuchungen über lineare Differentialgleichungen 4. Ordnung und die zugehörigen Gruppen.

VON SAUL EPSTEEN.

EINLEITUNG.

Lie hat versucht, seine Theorie der Transformationsgruppen, besonders der endlichen Gruppen (deren Transformationen nur von einer endlichen Anzahl Parameter abhängen) auf die Integration der Differentialgleichungen anzuwenden. Er zeigte, dass in fast allen Fällen, wo es bisher gelungen war die Ordnung einer gewöhnlichen Differentialgleichung zu erniedrigen, der innere Grund dafür die Existenz von Transformationen mit einer endlichen Anzahl Parameter ist, welche die betreffende Differentialgleichung invariant lassen. Diese Transformationen bilden notwendig eine Gruppe. Jeder Transformationsgruppe, definirt durch ihre infinitesimalen Transformationen, ordnete er gewisse Funktionen zu (deren Wichtigkeit schon erkannt war), die Differentialinvarianten, welche ungeändert bleiben bei allen Transformationen der Gruppe, und nur bei diesen. Ausgenommen gewisse spezielle Fälle, auf welche wir hier nicht einzugehen brauchen, ist jede Gleichung, welche bei allen Transformationen der Gruppe ungeändert bleibt, eine Beziehung zwischen diesen Differentialvarianten.

So fruchtbar und weittragend seine Methoden sind, in diesem Falle reichen sie nicht aus, denn im Allgemeinen bleibt nicht jede gewöhnliche Differentialgleichung höherer Ordnung

$$f\left(x, y, \frac{dy}{dx} \cdots\cdot\right) = 0 \tag{1}$$

invariant gegenüber einer Gruppe im Sinne von Lie. Die partielle Differentialgleichung

$$\frac{\delta f}{\delta x} + p_1 \frac{\delta f}{\delta x_1} + p_2 \frac{\delta f}{\delta x_2} \cdots\cdot + p_n \frac{\delta f}{\delta x_n} = 0 \tag{2}$$

und die vollständigen Systeme die Lie untersucht hat, bleiben in der Tat invariant gegenüber einer Gruppe die aus einer gewissen Anzahl Parameter und den $n+1$ Variablen $x, x_1, x_2, \ldots . x_n$ aufgebaut ist. Aber solche Systeme sind von sehr spezieller Natur.

Im Vergleich zur Vollständigkeit der Galois'schen Theorie der algebraischen Gleichungen kann die Unvollständigkeit von Lie's Theorie am besten eingesehen werden wenn man bedenkt, dass:

1) Bei einer gegebenen Differentialgleichung man nicht immer behaupten kann, dass die Reduktion vermittelst der Gruppe die einzig mögliche ist.

2) Gewisse Gleichungen, wie die Differentialgleichung der geodätischen Linien der Flächen zweiter Ordnung, integriert werden können trotzdem sie keine Transformationen *in sich* gestatten.

Der Weg zur Verallgemeinerung von Lie's Methoden nach der Richtung der Galois'schen Theorie hin wurde zuerst von Picard angedeutet (Comptes Rendus, 1883, und Annales de Toulouse, 1887), und die erste Ausführung gab E. Vessiot (Paris, Thèse, 1892; Annales de l'École Normale Supérieure, 1892). Vessiot untersuchte besonders die Differentialgleichung zweiter Ordnung und einige spezielle Fälle der Differentialgleichung dritter Ordnung.

Die lineare Differentialgleichung vierter Ordnung, deren Coefficienten Funktionen von x sind

$$\frac{d^4y}{dx^4} + 4\lambda_1 \frac{d^3y}{dx^3} + 6\lambda_2 \frac{d^2y}{dx^2} + 4\lambda_3 \frac{dy}{dx} + \lambda_4 y = 0 \tag{3}$$

bietet ein ausgedehntes und interessantes Gebiet für die Untersuchung, welches bis jetzt noch sehr wenig betreten worden ist. Es ist für die vorliegende Arbeit viel zu weit, und ich kann hier nur den Grund zu einer eingehenderen Untersuchung legen.

Die sieben ersten Paragraphen sind nur der Vorbereitung gewidmet und notwendig noch unvollständig. §1 bringt einen Abriss der Picard-Vessiot'schen Theorie der Gleichung (3). Nicht nur ist derselbe notwendig für das richtige Verständniss des Folgenden, sondern er hat auch den Vorteil, eine vielleicht nicht ganz einfache Theorie an einem speziellen Beispiel zu erläutern. §§2–7 sind den Problemen gewidmet, die sich uns in §1 aufgedrängt haben. §§8–10 sind Anwendungen.

I. KAPITEL.

Ueber die rationale Integration linearer Differentialgleichungen 4. Ordnung.

§1.—*Ueber rationale Integration. Stellung der Aufgabe.*

Wir beschränken unsere Betrachtungen auf lineare homogene Differentialgleichungen 4. Ordnung.

$$f(y) \equiv \frac{d^4y}{dx^4} + 4\lambda_1 \frac{d^3y}{dx^3} + 6\lambda_2 \frac{d^2y}{dx^2} + 4\lambda_3 \frac{dy}{dx} + \lambda_4 y = 0.$$

Die Theorie der rationalen Integration ist eine Ausdehnung der Galois'schen Methode auf Differentialgleichungen.

Rationalitätsbereich: Wir definiren zunächst folgendermassen den Rationalitätsbereich.

$$\overbrace{\hspace{20em}}^{[R]}$$

Basis.	Operationen.
1. Alle Constanten.	1. Die rationalen algebraischen Operationen.
2. Die unabhängige Variable x.	2. Differentiation.
3. Unbestimmte Funktionen $y_1 \ldots . y_4$ (die später ein Fundamentalsystem von Lösungen bilden sollen).	
4. Die Coefficienten der Gleichung (1).	
5. Alle gegebenen Funktionen.	

Wenn wir einen Rationalitätsbereich $[R]$ durch einen andern $[R']$ erweitern in solcher Weise, dass zur Basis von $[R]$ eine gewisse Anzahl von Funktionen hinzugefügt werden, so sagen wir dass diese Funktionen *adjungiert* sind zu den Funktionen des Bereiches $[R]$.

Die Gleichung heisst *reducibel* wenn sie ein Integral gemein hat mit einer Differentialgleichung niedrigerer Ordnung deren Coefficienten dem Rationalitätsbereich angehören. Im andern Falle heisst sie *irreducibel.*

Eine rationale Differentialfunktion V soll im Gebiet $[R]$ definiert sein wie in §§8, 9, 10. Wenn wir für die Integrale $y_1 \ldots . y_4$ ein *ganz bestimmtes* Fundamentalsystem $y_1^0 \ldots . y_4^0$ setzen, dann nimmt V seinen *numerischen Wert* $V_0(x)$ an; V_0 ist nicht notwendig in $[R]$ enthalten.

Formale und numerische Invarianz: Führen wir auf die y die lineare Substitution von 4^2 Parameter aus,

$$y'_i = \sum_{k=1}^{4} a_{ik} y_k \qquad (i = 1 \ldots\ldots 4) \tag{2}$$

so geht V über in V' dessen numerischer Wert V'_0 ist. Wenn V und V' identisch sind (§3) sagt Klein, dass V *formal invariant* ist; wenn aber $V'_0 \equiv V_0$ so ist V_0 *numerisch invariant* (Vorl. über höhere Geom., II, p. 299). Das Eine folgt nicht notwendig aus dem Andern.

Wenn $y_1 \ldots\ldots y_4$ ein Fundamentalsystem der Gleichung (1) bilden, so können wir offenbar schreiben

$$\frac{4.3 \ldots\ldots (4-k+1)}{1.2 \ldots\ldots k} \lambda_k = -\frac{D_k}{D} \qquad (k = 1 \ldots\ldots 4) \tag{3}$$

wo

$$D = \begin{vmatrix} y_1 & \dfrac{dy_1}{dx} & \ldots\ldots & \dfrac{d^3 y_1}{dx^3} \\ \vdots & & & \vdots \\ y_4 & \dfrac{dy_4}{dx} & \ldots\ldots & \dfrac{d^3 y_4}{dx^3} \end{vmatrix}$$

und

$$D_k = \begin{vmatrix} y_1 & \ldots\ldots & \dfrac{d^{k-1} y_1}{dx^{k-1}} & \dfrac{d^{k+1} y_1}{dx^{k+1}} & \ldots\ldots & \dfrac{d^4 y_1}{dx^4} \\ \vdots & & \vdots & & & \vdots \\ y_4 & \ldots\ldots & \dfrac{d^{k-1} y_4}{dx^{k-1}} & \dfrac{d^{k+1} y_4}{dx^{k+1}} & \ldots\ldots & \dfrac{d^4 y_4}{dx^4} \end{vmatrix}.$$

Die Gruppe (2) spielt in der Gleichung (1) die nämliche Rolle wie die Gruppe der Substitutionen von vier Buchstaben in der Theorie der algebraischen Gleichungen 4. Grades.

Es möge nun $\Omega\,(y_1 \ldots\ldots y_4)$ eine rationale Funktion von den Integralen von (1) und ihren Ableitungen nach x sein; dann nennen wir der Kürze halber Ω eine "rationale Funktion der Integrale" (Gl. 2, §8). Die Funktionen Ω welche invariant bleiben bei allen Transformationen (4) spielen hier dieselbe Rolle wie die symmetrischen Funktionen der Wurzeln in der algebraischen Theorie. In Wirklichkeit sind diese Ω Functionen der λ, welche wir "fundamentale Invarianten" nennen; denn Appell hat gezeigt (Annales de l'École Supérieure, 1881)

dass alle rationalen Funktionen von $y_1 \ldots\ldots y_4$ rational ausgedrückt werden können durch $x, \lambda_1, \ldots\ldots \lambda_4$ und ihre Ableitungen.

Gruppe der invarianten Funktion. Die Transformationen der Gruppe (2) welche Ω zulässt bilden eine Untergruppe, genannt die Gruppe der Funktion Ω. Man erhält die endlichen Transformationen in der Gleichung

$$\Omega\,(y_1' y_2' y_3' y_4') \equiv \Omega\,(y_1 y_2 y_3 y_4). \tag{4}$$

Dieselbe gibt uns eine gewisse Anzahl von Beziehungen zwischen den Constanten a_{ik}, aus welchen wir ihre Ausdrücke durch eine bestimmte Anzahl von ihnen erhalten (§6).

Transformierte Gleichung. Im Fall die Gruppe von Ω, $\rho = 4^2 - s$ Parameter enthält, d. h. dass Ω ρ unabhängige lineare homogene Transformationen zulässt, dann ist Ω, betrachtet als Funktion von x, das Integral einer Differentialgleichung ρ-ter Ordnung ($\rho < 16$). Dieselbe heisst *die Transformierte* der Gleichung (1). Wir schreiben sie

$$\Psi\left(F_1 \frac{dF}{dx} \ldots\ldots \frac{d^sF}{dx^s}\right) = 0, \tag{5}$$

wo $F = \Omega\,(y_1' \ldots\ldots y_4')$.

So ist z. B. $F = y_2^3 - y_3^2 y_4$ invariant gegenüber der vier-parameter Gruppe IX von §3 und die transformierte Gleichung von (1) welche F definiert, ist von der Ordnung $\rho = 4^2 - 4 = 12$, und kann erhalten werden durch zwölfmalige Differentiation von F und nachfolgende Elimination.

Die Gleichung (5) hat als Integrale Ω und die Werte welche man erhält, wenn mann an Ω alle Substitutionen der Gruppe ausübt—sie hat keine anderen Integrale.

Resolventen: Lehrsatz: Wenn eine rationale Funktion der Integrale $y_1 \ldots y_4$ einer linearen Differentialgleichung 4. Ordnung keine lineare homogene Transformation in $y_1 \ldots\ldots y_4$ gestattet, so können diese Integrale vermittelst dieser Funktion der Coefficienten der Gleichung und ihrer Ableitungen nach x ausgedrückt werden. Eine solche Funktion ist z. B. $V = u_1 y_1 + u_2 y_2 + \ldots\ldots u_4 y_4$ wo die u alle Funktionen von x sind.

Analogon zu Lagrange's Theorem: Wenn eine rationale Funktion $R\,(y_1 \ldots y_4)$ alle Transformationen der Gruppen der Funktionen von $\Omega_1 \ldots\ldots \Omega_p$ zulässt, kann dieselbe vermittels dieser Funktionen, der Coefficienten der Gleichung und ihrer Ableitungen nach x rational ausgedrückt werden.

So gestattet die Funktion (3) §9 alle die Transformationen von (1) und (2), und ist folglich eine rationale Funktion der beiden Letzteren, von $\lambda_1 \ldots\ldots \lambda_4$ und ihren Ableitungen in Anbetracht von x.

Bei Anwendung dieses Lehrsatzes lässt sich ersehen, dass die Gleichung (5) wenn irreducibel, die Eigentümlichkeit besitzt, dass ihr allgemeines Integral als eine algebraische Funktion eines besonderen Integrals durch eine Formel ausgedrückt werden kann, welche unverändert bleibt, was für ein Integral wir auch wählen. Die Gleichung (5) entspricht folglich den Abel'schen Gleichungen der algebraischen Theorie.

Rationalitätsgruppe (*Transformationsgruppe*): Es ist klar, dass im Allgemeinen keine Differentialfunktion V einen rationalen Wert besitzt. Hat sie doch einen solchen, so heisst die Gleichung eine *specielle Gleichung* in $[R]$. Zur genauen Untersuchung dieser speziellen Fälle bedienen wir uns der folgenden Lehrsätze von Picard, ergänzt durch Vessiot:

Dieser Gleichung entspricht in Bezug auf den Rationalitätsbereich $[R]$, eine Untergruppe Γ der linearen homogenen Gruppe (2) mit folgenden Eigenschaften:

1) Jede rationale Differentialfunktion V, deren numerischer Wert rational ist gestattet numerisch alle Transformationen von Γ.

2) Jede Funktion V, welche numerisch alle Transformationen von Γ zulässt, besitzt einen rationalen numerischen Wert.

Γ wird die *Rationalitätsgruppe* von (1) oder die *Transformationsgruppe* der Gleichung (siehe §2 im letzten Abschnitt) genannt. Für eine specielle Gleichung ist diese Gruppe charakteristisch, und da das Fundamentalsystem der Integrale willkürlich gewählt werden kann, folgt daraus, dass wir nur die *Typen* der Gruppen in Betracht zu ziehen brauchen.

Charakteristische Invarianten Typus der Gleichung: Jede spezielle Gleichung wird durch eine Verbindung der Form $\Omega(x, y_1, y_2, \ldots\ldots, y_1' \ldots\ldots) = a(x)$, charakterisiert wo $y_1 \ldots\ldots y_4$ ein System von Fundamentalintegralen bilden. Ω gestattet die Transformationen von Γ entweder formal oder numerisch, jedoch keine andere. $a(x)$ gehört zu $[R]$. Ω nennt eine *charakteristische Invariante von* Γ. Um den *Typus* festzustellen, welchem eine gegebene lineare Differentialgleichung angehört, is es desshalb notwendig, folgende Probleme zu lösen:

1) Bestimme die verschiedenen Typen der linearen homogenen Gruppen in vier Variabeln. (In §3 finden sich unter andern interessanten Gruppen gewisse *primitive* Typen.)

2) Bestimme Ω für jede dieser Typen und bilde die Differentialgleichung von welcher Ω abhängt [die Transformierte der Gleichung (1)].

3) Der gesuchte Typus wird die kleinste Gruppe sein die einer transformierten Gleichung entspricht, welche ein Integral besitzt das rational in x ist.

Angenommen Γ_1 sei die grösste ausgezeichnete Untergruppe von Γ und Ω_1 eine charakteristische Invariante von Γ_1, so wird Ω_1 als eine Funktion von Ω betrachtet, einer gewissen Differentialgleichung genügen. Integriert man diese Gleichung, und adjungiert man eines dieser Integrale, so wird die Transformationsgruppe auf Γ_1 reduziert. Durch dieselbe Prozedur kann man sie noch mehr reduzieren. Der Wert dieser Methode liegt in folgendem Lehrsatz von Vessiot (von Picard verallgemeinert): Wenn die vollständige Integration einer rationalen Hülfsgleichung die Gruppe Γ reduziert, dann ist sie zu einer invarianten Untergruppe reduziert (§7).

In besonderen Fällen, wenn die Hülfsgleichung der ersten Ordnung angehört, wird die Gruppe zu einer invarianten Untergruppe mit einem Glied weniger reduziert.

Ueberblick: Kurz, diese Integrationstheorie hängt von der Lösung folgender Probleme ab:

I. Feststellung der verschiedenen Typen der algebraischen Gruppen (§6) welche in der linearen homogenen Gruppe der vier Variablen enthalten sind.

II. Feststellung der charakteristischen Bedingungen unter welchen die Transformationsgruppe der gegebenen linearen Gleichung auf einen dieser Typen reduziert wird, [§8 Gl. (3'), §9 Gl. (11)].

III. Gib, wenn der Typus bekannt ist, die Natur der Hülfsgleichungen an, welche zu integrieren sind.

Diesen drei Problemen fügen wir das viel schwierigere bei:*

IV. Die Differentialgleichung ist gegeben, bestimme ihre Rationalitätsgruppe.

§2.—*Zergliederung des Problems.*

Integrabele Gruppen. Unter einer *integrabelen Gruppe* versteht man eine Gruppe welche eine invariante Untergruppe, mit einem Parameter weniger als

* Vgl. Epsteen, "Determination of the Group of Rationality of a Linear Differential Equation," American Mathematical Monthly, January, 1903, p. 4.

die gegebene Gruppe, enthält; diese Untergruppe hat wieder eine invariante Untergruppe mit einem Parameter weniger als sie selbst, u. s. w.

Im 2. Teil, Kapitel VI seiner berühmten Thesen von 1892 hat Vessiot die Hauptbedingungen der Integrabilität durch Quadraturen angegeben. Obgleich an einem gegebenen Beispiel wie unserm, solche Untersuchungen gewöhnlich detaillierter ausgeführt werden können, unterlassen wir es jetzt und beschränken uns auf das Studium der nicht integrablen Untergruppen der linearen homogenen Gruppen (§1, 1) und ganz besonders derjenigen, welche algebraisch sind (§1, I; §6). Dazu bedienen wir uns des Engel'schen Lehrsatzes:

Bedingungen für die Integrabilität. Eine Gruppe ist stets dann, aber auch nur dann integrabel, wenn sie keine dreigliedrige Untergruppe, mit der Structur der allgemeinen projectiven Gruppe mit einer Variabeln, enthält.

Nach unserer Methode haben wir daher vor allem zu bestimmen, welche der Gruppen von §3 Untergruppen haben, die isomorph mit der projectiven Gruppe mit einer Variabeln und drei Parameter sind.

Wir wollen daher die Gruppen, welche Untergruppen dieser Art enthalten, herausgreifen und annehmen, dass dieses die Transformationsgruppe der Differentialgleichung sei. Auf diese Weise werden wir das in §1, II vorgeschlagene Problem lösen. Dann wird die Art der zu integrierenden Hülfsgleichungen offenbar werden.

II. KAPITEL.

Ueber die Gruppen in R_4.

§3.—*Ueber die linearen homogenen Gruppen in R_4.*

Wenn alle Gruppen in R_4 bekannt sind. Die infinitesimalen Transformationen der linearen homogenen Gruppe werden gebildet aus Combinationen von

$$x_i p_k \qquad (i, k = 1 \ldots . 4)$$

indem man x statt y gebraucht um es in Uebereinstimmung mit der gewöhnlichen Schreibweise zu bringen.

Wenn wir dann die Gruppen in R_4 gegeben haben, brauchen wir nur die Glieder $x_i p_k$ auszuwählen und *nachzusehen, ob diese für sich eine Gruppe bilden.* Da infolge ihrer grossen Anzahl und der Länge ihrer Berechnung nach nicht

alle Gruppen in R_4 aufgezählt worden sind, können wir uns nicht vollständig auf diese sonst leichte Methode verlassen.

Alle projektiven Gruppen in R_3 sind bekannt. Nach einer andern Methode nimmt man die projektiven Gruppen in R_3, welche beinahe alle bekannt sind, und verwandelt sie in lineare homogene Gruppen in R_4, indem man folgende Uebergangsformeln Lie's anwendet (Transformationsgruppen I, p. 579):

$$\left.\begin{array}{lll}
p \equiv x_4 p_1 & q \equiv x_4 p_2 & r \equiv x_4 p_3 \\
xp \equiv x_1 p_1 - \frac{1}{4}\sum_{i=1}^{4} x_i p_i & xq \equiv x_1 p_2 & xr \equiv x_1 p_3 \\
yp \equiv x_2 p_1 & yq \equiv x_2 p_2 - \frac{1}{4}\sum_{i=1}^{4} x_i p_i & yr \equiv x_2 p_3 \\
zp \equiv x_3 p_1 & zq \equiv x_3 p_2 & zr \equiv x_3 p_3 - \frac{1}{4}\sum_{i=1}^{4} x_i p_i \\
xU \equiv -x_1 p_4 & yU \equiv -x_2 p_4 & zU \equiv -x_3 p_4 \\
U + xp \equiv x_1 p_1 - x_4 p_4 & U + yq \equiv x_2 p_2 - x_4 p_4 & U + zr \equiv x_3 p_3 - x_4 p_4
\end{array}\right\} (1)$$

wo $U \equiv xp + yq + zr$ ist.

Ich sagte, dass die projektiven Gruppen in R_3 *beinahe* alle bekannt seien, denn Lie und seine Schüler haben eine Anzahl derselben ausgerechnet und die Methoden angegeben, die zu deren vollständigen Feststellung anzuwenden sind; aber so viel ich weiss ist das noch nicht im Einzelnen durchgeführt worden.

Diese Methode befolgt man in den meisten Fällen in denen die linearen homogenen Gruppen in R_4 zu bestimmen sind (indem wir uns der Resultate der "Transformationsgruppen" III, Abteilung III bedienen). Bei den primitiven Gruppen in R_4 können wir die erste Methode anwenden, da Lie zwei dieser Gruppen bestimmt hat und die übrigen neun von J. M. Page (American Journ., 1888, Leipz. Dissert., 1888) ausgearbeitet worden sind.

Gruppen in R_4: Die Gruppen, welche die Richtungen eines Punktes von allgemeiner Lage in R_4, der fest bleibt, transformiren, sind I, II und III:

I $\boxed{p_k;\ x_i p_k;\ x_1 U;\ x_2 U;\ x_3 U;\ x_4 U; \qquad (i, k = 1 \ldots . 4)}$

Wie gebräuchlich ist $U = \sum_{i=1}^{4} x_i p_i$. Dieses ist die allgemeine projektive Gruppe in R_4.

II $\boxed{p_k,\ x_i p_k, \qquad (i, k = 1 \ldots . 4)}$

die gegebene Gruppe, enthält; diese Untergruppe hat wieder eine invariante Untergruppe mit einem Parameter weniger als sie selbst, u. s. w.

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wo
$$U \equiv xp + yq + zr \text{ ist.}$$

Ich sagte, dass die projektiven Gruppen in R_3 *beinahe* alle bekannt seien, denn Lie und seine Schüler haben eine Anzahl derselben ausgerechnet und die Methoden angegeben, die zu deren vollständigen Feststellung anzuwenden sind; aber so viel ich weiss ist das noch nicht im Einzelnen durchgeführt worden.

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Wie gebräuchlich ist $U = \sum_{i=1}^{4} x_i p_i$. Dieses ist die allgemeine projektive Gruppe in R_4.

II $$\boxed{p_k,\ x_i p_k, \qquad (i, k = 1 \ldots. 4)}$$

Dieses ist die allgemeine lineare Gruppe.

III $$\boxed{p_k, x_i p_k; x_i p_i - x_k p_k \qquad (i, k = 1 \ldots . 4) \quad i \neq k}$$

Dieses ist die spezielle lineare Gruppe.

Die einzigen Glieder in I und II, die wir gebrauchen können, sind:

$$x_i p_k, \qquad (i, k = 1 \ldots . 4). \tag{2}$$

Bilden diese für sich allein eine Gruppe? Bilde den Klammerausdruck von Jacobi:

$$[x_i p_k, x_{i'} p_{k'}].$$

Nun ist $p_k(x_{i'}) = 0$ oder 1 je nachdem $k \neq i'$ oder $k = i'$ ist. Ebenso ist ${}_{k'}(x_i) = 0$ oder 1 je nachdem $k' \neq i$ oder $k_1 = i$ ist.

Ein wenig Nachdenken führt uns zu der Ueberzeugung, dass (2) in der That eine Gruppe bildet. Wir haben festzustellen, ob diese Gruppe integrabel ist oder nicht.

Dass (2) eine Gruppe bildet liesse sich schon daraus schliessen, dass II aus zwei Arten Glieder besteht, p_k—Translationen und $x_i p_k$—Rotationen; es ist klar, dass eine jede derselben für sich allein eine Gruppe bildet, doch kann diese Methode nicht immer angewendet werden—so z. B. nicht bei I—und ist es desshalb besser die allgemeine Methode zu gebrauchen, welche sich auf alle Fälle anwenden lässt.

Die einzigen Glieder in III enthalten, welcher wir uns bedienen können, sind:

$$\left.\begin{array}{ll} x_i p_k, & x_i p_i - x_k p_k, \\ i \neq k, & (ik = 1 \ldots . 4). \end{array}\right\} \tag{3}$$

Da $[x_i p_k, x_i p_i - x_k p_k] = -2x_i p_k$ ist, folgt daraus, dass die Glieder (4) eine Gruppe bilden. Wir haben festzustellen, ob diese integrabel ist oder nicht.

Die Gruppen, welche eine Fläche zweiter Ordnung invariant lassen, sind: IV, V, VI, VII und die imprimitive Gruppe VIII.

IV $$\boxed{p_i, x_i p_k - x_k p_i \qquad (i, k = 1 \ldots . 4)}$$

Dieses ist die Gruppe der Euklidischen Bewegungen und Aehnlichkeitstransformationen.

V $\boxed{p\,,\ x_i p_k - x_k p_i,\ U \qquad (i, k = 1 \ldots . 4)}$

VI $\boxed{p_i - x_i U,\ x_i p_k - x_k p_i \qquad (i, k = 1 \ldots . 4)}$

Diese Gruppe lässt $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ invariant.

VII $\boxed{p_i,\ x_i p_k - x_k p_i,\ U,\ 2x_i U - p_i \sum_{j=1}^{4} x_j^2 \qquad (i, k = 1 \ldots . 4)}$

Dieses ist die Gruppe der conformen Transformationen.

Wir können ihnen die imprimitive Gruppe beifügen ;

VIII $$\boxed{\begin{array}{lll} x_4 p_1 + x_2 p_3, & x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4, & x_3 p_2 + x_1 p_4, \\ x_4 p_2 + x_1 p_3, & -x_1 p_1 + x_2 p_2 + x_3 p_3 - x_4 p_4, & x_3 p_1 + x_2 p_4\ U \end{array}}$$

welche $x_3 x_4 - x_1 x_2 = 0$ invariant lässt.

In IV und VI können wir noch folgende Glieder anwenden:

$$x_i p_k - x_k p_i \tag{4}$$

und in V und VII folgende:

$$x_i p_k - x_k p_i, \quad U. \tag{5}$$

Es ist sehr leicht zu beweisen, dass (5) eine Gruppe bildet, von welcher (4) eine Untergruppe ist. Wir müssen nach der Integrabilität dieser Gruppen forschen. Es ist nebenbei zu bemerken, dass wenn (4) nicht integrabel ist, (5) es auch nicht ist.

Dieselbe Untersuchung muss mit VIII vorgenommen werden.

Die Gruppen, welche eine Curve dritter Ordnung wie

$$\left.\begin{array}{l} x_2 x_4 - x_1^2 = 0, \\ x_3 x_4^2 - x_1^3 = 0, \end{array}\right\} \tag{6}$$

in R_3 invariant lassen, sind IX und X.

IX $\boxed{\begin{array}{c} p_k,\ x_1 p_1 - x_2 p_2 + 3\,(x_4 p_4 - x_3 p_3),\ x_4 p_1 + 2x_1 p_2 + 3x_2 p_3, \\ 2x_2 p_1 + x_3 p_2 + 3x_1 p_4,\ U \end{array}}$

und deren Untergruppe

X $\boxed{\begin{array}{c} p_k,\ x_1 p_1 - x_2 p_2 + 3\,(x_4 p_4 - x_3 p_3),\ x_4 p_1 + 2x_1 p_2 + 3x_2 p_3, \\ 2x_2 p_1,\ + x_3 p_2 + 3x_1 p_4. \end{array}}$

Hier brauchen wir nur die Glieder p_k wegzulassen und was übrig bleibt, bildet augenscheinlich eine Gruppe. Was von X übrig bleibt, ist schon eine Gruppe von drei Parametern und wir können gleich Engel's Lehrsatz (§2) darauf anwenden. Und da die Glieder von X mit U vertauschbar sind, folgt daraus, dass dieselben Schlüsse auf IX passen.

Die Gruppen, welche den linearen Complex

$$dz + xdy - ydx = 0 \tag{7}$$

invariant lassen, sind XI und XII.

XI $\boxed{\begin{array}{c} x_4 p_1 - x_2 p_3,\ x_4 p_2 + x_1 p_3,\ x_4 p_3,\ x_1 p_1 - x_2 p_2,\ x_2 p_1, \\ x_3 p_1 + x_2 p_4,\ x_3 p_2 - x_1 p_4,\ x_3 p_4,\ x_1 p_2,\ x_3 p_3 - x_4 p_4\ U \end{array}}$

und deren Untergruppe

XII $\boxed{\begin{array}{c} x_4 p_1 - x_2 p_3,\ x_4 p_2 + x_1 p_3,\ x_4 p_3,\ x_1 p_1 - x_2 p_2,\ x_2 p_1, \\ x_3 p_1 + x_2 p_4,\ x_3 p_2 - x_1 p_4,\ x_3 p_4,\ x_1 p_2,\ x_3 p_3 - x_4 p_4. \end{array}}$

Wie bevor, müssen wir die Integrabilität von XII erforschen und es lassen sich dieselben Folgerungen auf XI anwenden.

Ueberblick: Wenn wir uns kurz fassen, müssen wir in Betracht ziehen :

$\left.\begin{array}{l} \text{I} \\ \text{II} \end{array}\right\}$ die Glieder $\boxed{\quad x_i\, p_k \quad}$

III die Glieder $\boxed{x_i p_k,\ x_i p_i - x_k p_k \qquad i \neq k}$

IV, VI die Glieder $\boxed{x_i p_k - x_k p}$

V, VII die Glieder $\boxed{x_i p_k - x_k p_i,\ U}$

VIII alle Glieder.

IX, X alle Glieder ausser p_k.

XI, XII alle Glieder.

Diesen können wir noch die imprimitive Gruppe (Transformationsgruppen III, p. 214)

XIII $\boxed{\begin{array}{c} x_4 p_1,\ x_4 p_2,\ x_4 p_3,\ 2x_3 p_1 + x_2 p_3,\ x_1 p_1 - x_2 p_2, \\ 2x_3 p_2 + x_1 p_3,\ U \end{array}}$

beifügen, welche den unendlich fernen Kegelschnitt

$$x_4 = 0, \quad x_1 x_2 - x_3^2 = 0$$

invariant lässt.

Bemerkung: Es wird für später von Nutzen für uns sein, wenn wir uns merken, dass die invarianten Configurationen der angeführten Gruppen entweder zweiten oder dritten Grades sind.

§4.— *Ueber dreigliedrige Untergruppen.*

Methoden zur Bestimmung der dreigliedrigen Untergruppen. Der nächste Schritt wird sein, zu bestimmen, ob die Gruppen in §3 integrierbar sind oder nicht. Wenn die Gruppe integrierbar ist, so kann die Gleichung, welcher die Gruppe entspricht, mittels Quadraturen integriert werden, andern Falls ist die Gleichung nicht so einfach zu lösen. Die allgemeine Methode (§2) besteht darin, die dreigliedrigen Untergruppen der gegebenen Gruppen zu bestimmen und nachzusehen, ob sie isomorph sind mit der dreigliedrigen projectiven Gruppe mit einer Variabeln. Es gibt zwei Methoden für die Bestimmung:

1) Man nimmt die projectiven Gruppen im dreidimensionalen Raum und ihre Untergruppen, wie sie Lie in den Transformationsgruppen (III; Abteilung III) gegeben hat, und geht, indem man die Uebergangsformeln von §3 anwendet, über zu den linearen homogenen Gruppen und Untergruppen im vierdimensionalen Raum (R_4).

2) Man wendet die algebraische Methode an, wie sie Lie in den Transformationsgruppen (I, pag. 208) gegeben hat.

Die letztere Methode ist gewöhnlich so lang, dass man sie nur anwenden wird, wenn andere Methoden versagen.

Methoden der derivierten Gruppen: Weil unser letztes Ziel ist, zu bestimmen, welche von den Gruppen in R_4 nicht integrabel sind, und welche—wenn es überhaupt solche giebt—integrabel sind, so können wir vom folgenden Lehrsatz Gebrauch machen: *

Eine r-gliedrige Gruppe ist dann und nur dann integrabel, wenn ihre r^{te} derivierte Gruppe sich auf die Identität reduciert.

Der Vorteil dieser Methode liegt in der Thatsache dass, falls die Gruppe integrabel ist, die erste derivierte entweder $X_1 X_2 \ldots . X_{r-1}$ ist, oder in ihr enthalten ist. Diese Bedingung ist notwendig, aber nicht hinreichend. Daher können wir ohne weiteres sagen, dass, wenn alle Glieder $X_1 \ldots . X_r$ wieder erscheinen, die Gruppe nicht integrabel ist. Oder wenn nötig, so können wir das Verfahren wiederholen und finden, dass, obgleich in der ersten derivierten Gruppe ein Glied z. B. X_r fehlt, die zweite derivierte Gruppe alle Glieder enthält, welche die erste derivierte Gruppe schon enthielt, und dass daher die r^{te} derivierte Gruppe sich nicht auf die Identität reduzieren kann. Der Grund hiefür ist, dass die zweite derivierte Gruppe zwei Glieder weniger enthalten muss, als die gegebene Gruppe. Es wird kaum nötig sein, zu erwähnen, dass die Bezeichnung $X_1 \ldots . X_r$ keine feste ist, sondern dass irgend eine der infinitesimalen Transformationen mit irgend einem X_s bezeichnet werden kann, und wenn die successiven derivierten Gruppen gebildet sind, so können wir jederzeit die infinitesimalen Transformationen anders benennen.

Allgemeiner können wir sagen:

Wenn die $(r - k)^{te}$ derivierte Gruppe $(k < r)$ alle Glieder der $(r - k + 1)^{ten}$ derivierten enthält, so ist die gegebene Gruppe nicht integrabel.

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* Vergl. Lie-Scheffers.—Contin. Gr., pag. 548.

Wenn r gross ist, so wird diese Methode langwierig, besonders wenn $k > 1$ ist, und die Methode mit Hülfe der dreigliedrigen Untergruppen ist dann eleganter.

Kombinierte Methode: Im besonderen finden wir mittelst dieser kombinierten Methode einen ausgezeichneten Ersatz für die Normalisierung der dreigliedrigen Untergruppen, wo man nachsieht, ob sie die Struktur der dreigliedrigen projectiven Gruppe mit einer Variabeln haben. Mit Rücksicht auf die obige Bemerkung ist $r = 3$ und $k < 3$; daher brauchen wir nur die erste derivierte Gruppe der betreffenden dreigliedrigen Untergruppe zu bestimmen; wenn alle drei Glieder $X_1 X_2 X_3$ wieder erscheinen, ist die Gruppe sicher nicht integrabel. Um behaupten zu können, dass die Gruppe *integrabel* sei, müssen wir diese Methode auf alle dreigliedrigen Untergruppen der gegebenen Gruppe anwenden und zeigen, dass *keine* von ihnen *perfekt* ist.

§5.—*Ueber die Integrabilität der Gruppen.*

Wir wollen nun die Gruppen betrachten, die wir am Schlusse des §3 zusammengestellt haben.

1) Greifen wir die spezielle lineare Gruppe heraus

$$\boxed{x_i p_k, \quad x_i p_i - x_k p_k} \tag{1}$$

und betrachten nur die glieder

$$X_1 = x_1 p_2, \quad X_2 = x_2 p_1, \quad X_3 = x_1 p_1 - x_2 p_2$$

so können wir leicht mit Hülfe des Jacobi'schen Klammerausdruckes verifizieren, dass sie eine Untergruppe der gegebenen Gruppe bilden, und dass diese ihre eigene erste derivierte Gruppe ist, woraus dann der Satz folgt:

Die spezielle lineare homogene Gruppe in R_4 ist eine nicht integrabele Gruppe.

Wie wir bereits gezeigt haben, schliesst dieser Satz den Fall der allgemeinen linearen homogenen Gruppe in sich und entspricht der Bedingung am Ende des §4, wo bewiesen wurde, dass eine Gruppe sicher nicht integrabel ist, wenn sie eine nicht integrable Untergruppe enthält.

2) Weil

$$\boxed{x_i p_k - x_k p_i} \tag{2}$$

eine Untergruppe von

$$\boxed{x_i p_k - x_k p_i, \quad x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4} \tag{3}$$

ist, so folgt durch dieselbe Kette von Schlüssen, dass (5) nicht integrabel ist vorausgesetzt, dass (3) es nicht ist.

Wählen wir die Glieder:

$$X_1 = x_1 p_2 - x_2 p_1, \quad X_2 = x_1 p_4 - x_4 p_1, \quad X_3 = x_2 p_4 - x_4 p_2 \tag{4}$$

so haben wir eine dreigliedrige Untergruppe, und

$$(X_1 X_2) = -X_3, \quad (X_1 X_3) = X_2, \quad (X_2 X_3) = X_1 \tag{5}$$

d. h. ihre dritte derivierte Gruppe kann sich nicht auf die Identität reducieren, und daher ist diese dreigliedrige Untergruppe perfect, woraus folgt:

Die betrachteten primitiven Gruppen in R_4, welche eine Fläche zweiter Ordnung invariant lassen, sind nicht integrabel.

Unter den dreigliedrigen Untergruppen von VIII, §3 ist eine

$$X_1 = x_4 p_1 + x_2 p_3, \quad X_2 = x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4, \quad X_3 = x_3 p_2 + x_1 p_4. \tag{6}$$

Ihre Zusammensetzung ist:

$$(X_1 X_2) = X_3, \quad (X_1 X_3) = -X_2, \quad (X_2 X_3) = 2X_3 \tag{7}$$

und daher kann ihre derivierte Gruppe sich nicht auf die Identität reducieren, woraus folgt:

Die imprimitive Gruppe VIII, welche die Oberfläche $x_3 x_4 - x_1 x_2 = 0$ invariant lässt, ist nicht integrabel.

3) Genau dieselben Schlüsse, wie oben, zeigen dass, wenn X nicht integrabel ist, es auch IV nicht ist.

Bezeichnen wir

$$\left.\begin{aligned} -2X_2 &= x_1 p_1 - x_2 p_2 + 3\,(x_4 p_4 - x_3 p_3), \\ X_1 &= x_4 p_1 + 2x_1 p_2 + 3\; x_2 p_3, \\ X_3 &= 2x_1 p_1 + x_3 p_2 + 3\; x_1 p_4, \end{aligned}\right\} \tag{8}$$

so folgt: $$(X_1 X_2) = X_1, \quad (X_1 X_3) = 2X_2, \quad (X_2 X_3) = X_3 \tag{9}$$

welche wiederum die Structur der Projectiven Gruppe der Geraden ist. Daher der Satz:

Die Gruppen IX und X, welche die Raumcurve dritter Ordnung

$$x_2 x_4 - x_1^2 = 0, \quad x_3 x_4^2 - x_1^3 = 0$$

invariant lassen, sind nicht integrabel.

4) Die Untergruppen der Gruppen XI und XII, welche den linearen Complex $dz + xdy - ydx = 0$ invariant lassen, sind vollständig behandelt worden von Knothe.* Dreigliedrige Untergruppen existieren nur, wenn kein Punkt invariant bleibt (siehe auch Lie-Engel, III, pag. 295).

Wenn keine gerade Linie invariant bleibt, so bleibt eine Raumcurve dritter Ordnung invariant. Aber wir haben bereits (in No. 3) gesehen, dass in diesem Falle die Gruppe nicht integrabel ist.

Wenn eine gerade Linie invariant bleibt, so ist sie eine Complexgerade oder eine Nichtcomplexgerade.

Wenn sie eine Nichtcomplexgerade ist, so ist die grösste zugehörige Untergruppe von XI und XII:

$$\boxed{x_1 p_2, \quad x_1 p_1 - x_2 p_2, \quad x_2 p_1, \quad x_4 p_3, \quad x_3 p_3 - x_4 p_4, \quad x_3 p_4.} \tag{10}$$

Ihre Untergruppe

$$X_1 = x_4 p_3, \quad X_2 = x_3 p_3 - x_4 p_4, \quad X_3 = x_3 p_4 \tag{11}$$

hat die Zusammensetzung

$$(X_1 X_2) = 2X_1, \quad (X_1 X_3) = -X_2, \quad (2X_2 X_3) = 2X_3 \tag{12}$$

und ist daher nicht integrabel.

Wenn die gerade Linie eine Complexgerade ist, haben wir den Fall der Euklidischen Bewegungen und Aenlichkeitstransformationen, welche wir schon (in No. 2) betrachtet haben.

Wir schliessen daraus:

* E. Knothe, Archiv for Mathematik og Naturvidenskab, Bd. 15, 1892.

Die Gruppen XI, XII, welche den linearen Complex

$$dz + xdy - ydx = 0 \tag{13}$$

invariant lassen sind nicht integrabel.

5) Es bleibt noch die Gruppe XIII welche den unendlich fernen Kegelschnitt $x_4 = 0$, $x_1 x_2 - x_3^2 = 0$ invariant lässt.

Unter anderen enthält sie die dreigliedrige Untergruppe

$$X_1 = 2x_3 p_1 + x_2 p_3, \quad X_2 = x_1 p_1 - x_2 p_2, \quad -X_3 = 2x_3 p_2 + x_1 p_3 \tag{14}$$

von der man sofort erkennt, dass sie, wie folgt zusammengesetzt ist:

$$(X_1 X_2) = X_1, \quad (X_1 X_3) = 2X_2, \quad (X_2 X_3) = X_3$$

und, wie oben, folgt dann:

Die Gruppe XIII, welche den unendlich fernen Kegelschnitt

$$x_4 = 0, \quad x_1 x_2 - x_3^2 = 0$$

invariant lässt, ist nicht integrabel.

Alles zusammengefasst haben wir also:]

Alle Gruppen in der Zusammenstellung in §3 *sind nicht integrabel.*

Es ist vielleicht von Interesse, einen Ausnahmsfall zu den obigen Gruppen auszuführen, nämlich eine integrabele Gruppe.

Betrachten wir die Gruppe:

$$\text{XIV} \qquad \boxed{\begin{array}{c} X_1 = x_4 p_1 + 2x_1 p_2 + 3x_2 p_3, \quad X_2 = x_1 p_1 + 2x_2 p_2 + 3x_3 p_3, \\ X_3 = x_4 p_2 + 3x_1 p_3, \quad X_4 = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 \end{array}}$$

welche die Cayley'sche Oberfläche

$$3x_1 x_2 x_4 - x_3 x_4^2 - 2x_1^3 = 0 \tag{16}$$

invariant lässt.

Die Gruppe $X_1 X_2 X_3 X_4$ enthält $X_1 X_2 X_3$ als eine invariante Untergruppe, und diese Gruppe ihrerseits die invariante Untergruppe $X_2 X_3$ und daher gilt:

Die Gruppe, welche die Cayley'sche Oberfläche dritter Ordnung

$$3x_1 x_2 x_4 - x_3 x_4^2 - 2x_1^3 = 0 \tag{17}$$

invariant lässt, ist integrabel.

§6.—*Beweis dass die Gruppen algebraisch sind:*

Erste Methode: Nach dem, was wir in §2 angeführt haben, haben wir uns nur auf algebraische Gruppen zu beschränken.

Die Gruppen in §3 sind homogen, aber weil wir uns ausschliesslich nur mit infinitesimalen Transformationen beschäftigt haben, so müssen wir uns überzeugen, dass sie wirklich algebraische Gruppen darstellen.

Es ist unnötig, diese Untersuchung für alle Gruppen des §3 zu machen, und wir wollen andeuten, wie wir uns von dieser Thatsache überzeugen können.

Nach den wohlbekannten Lie'schen Methoden (Continuierliche Gruppen, Kapitel 7) können wir die endlichen Gleichungen der Gruppen thatsächlich ausrechnen, und uns überzeugen dass sie algebraisch sind.

Zweite Methode: In §1, Gl. (4) haben wir eine angegeben, um die endlichen Transformationen zu finden. Diese Methode können wir hier sehr gut anwenden, da wir ja in jedem Falle das invariante Gebilde kennen.

Betrachten wir als Beispiel der Untergruppe von VIII, indem wir das Glied U weglassen. Sie stellt eine sechsgliedrige Gruppe vor, welche die Oberfläche $x_3 x_4 - x_1 x_2 = 0$ invariant lässt.

Um die endlichen Gleichungen dieser Gruppe zu bestimmen, schreiben wir:

$$x_3' x_4' - x_1' x_2' \equiv x_3 x_4 - x_1 x_2, \tag{1}$$

d. h. ausführlich:

$$\begin{aligned}(a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + a_{34} x_4)(a_{41} x_1 + a_{42} x_2 + a_{43} x_3 + a_{44} x_4)&\\ -(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4)(a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4) &\equiv x_3x_4 - x_4x_2.\end{aligned} \tag{2}$$

Dies giebt uns zehn Gleichungen zwischen den sechszehn Grössen a, und wir können daher zehn von ihnen als Funktionen von den sechs übrigen als unabhängigen Variabeln bestimmen, was beweist dass unsere Gruppe sechs unabhängige Parameter enthält.

Es ist aber begreiflich, dass Fälle vorkommen können, wo es unmöglich ist, eine gewisse Anzahl der Parameter, sagen wir r, als Funktionen der übrigen $16 - r$ darzustellen; aber diese Unmöglichkeit kann hier nicht vorkommen, wie die andere Methode zeigt.

Daraus folgt:

Wir können die endlichen Gleichungen unserer Gruppen bestimmen, indem wir simultane algebraische Gleichungen auflösen.

§7.—*Invariante Untergruppen.*

Gemäss unserer allgemeinen Theorie reduciert die Adjungierung der Integrale einer Hülfsgleichung die Transformationsgruppe zu einer invarianten Untergruppe (siehe §1). Die Kenntniss der invarianten Untergruppen ist daher notwendig zu vollständigem Verständniss der Natur der auszuführenden Reductionen.

Unter einer m-gliedrigen Untergruppe der r-gliedrigen Gruppe $X_1 \ldots\ldots X_r$ verstehen wir, dass

$$[X_i X_k] = X_s, \qquad i, k, s = 1 \ldots\ldots m. \tag{1}$$

Die Untergruppe ist invariant, wenn

$$[X_i X_k] = X_s, \qquad \begin{cases} i, s = 1 \ldots\ldots m, \\ \quad k = 1 \ldots\ldots r. \end{cases} \tag{2}$$

Die allgemeine lineare Gruppe enthält als invariante Untergruppe die specielle lineare Gruppe, die letztere keine invariante Untergruppe.

V hat IV zur invarianten Untergruppe, während die letztere keine invariante Untergruppe hat. Wenn die Anzahl der Parameter klein ist, wie meist in diesen Fällen, so können wir, in Ermangelung anderer Methoden, alle möglichen linearen Combinationen der Transformationen bilden (in diesem Falle alle möglichen linearen Combinationen von 5, 4, 3, 2 Gliedern), dann nachsehen ob sie, gemäss (1), Untergruppen bilden; und, wenn dies der Fall, so zeigt (2), wie man findet, ob sie invariante Untergruppen sind.

VIII hat als invariante Untergruppen:

$$\boxed{x_4 p_1 + x_2 p_3, \quad x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4, \quad x_3 p_2 + x_1 p_4} \tag{3}$$

und

$$\boxed{x_4 p_2 + x_1 p_3, \quad -x_1 p_1 + x_2 p_2 + x_3 p_3 - x_4 p_4, \quad x_3 p_1 + x_2 p_4} \tag{4}$$

Diese Tatsache ist auch leicht zu verificieren, da ja VIII dieselbe Structur hat, wie die projective Gruppe:

$$\boxed{p, \quad xp, \quad x^2 p, \quad q, \quad yq, \quad y^2 q} \tag{5}$$

Die Gruppe IX hat X als invariante Untergruppe, und X hat keine invariante Untergruppe, weil sie genau dreigliedrig ist, und wir gesehen haben, dass sie nicht integrabel ist.

XIII hat als invariante Untergruppen:

$$\boxed{x_4p_1,\ x_4p_2,\ x_4p_3,\ 2x_3p_1+x_2p_3,\ x_1p_1-x_2p_2,\ 2x_3p_2+x_2p_3} \tag{6}$$

$$\boxed{x_4p_1,\ x_4p_2,\ x_4p_3,\ x_1p_1+x_2p_2+x_3p_3+x_4p_4} \tag{7}$$

$$\boxed{x_4p_1,\ x_4p_2,\ x_4p_3} \tag{8}$$

XI hat XII als invariante Untergruppe.

Endlich enthält XIV, weil sie integrabel ist, eine dreigliedrige invariante Untergruppe, und die letztere eine zweigliedrige invariante Untergruppe.

III. KAPITEL.

Anwendungen.

Einleitung. Wir müssen nun gemäss §1 voraussetzen, dass jede Gruppe der Reihe nach die Transformationsgruppe unserer Differentialgleichung ist und jetzt sehen wir mit Hülfe unserer gruppen-theoretischen Untersuchungen, dass wir drei *Arten* von Problemen vor uns haben, welche wir folgendermassen formulieren können:

1) Die Reduction zu untersuchen, die aus der Existenz der speciellen linearen Gruppe als eine invariante Untergruppe der allgemeinen linearen Gruppe folgen.

2) Die Eigenschaften unserer Differentialgleichung zu untersuchen, wenn eine cubische Relation zwischen den Integralen invariant bleibt.

3) Die Eigenschaften unserer Differentialgleichung zu untersuchen, wenn eine quadratische Relation zwischen den Integralen invariant bleibt.

§8.—*Erste Reduction, die sich aus der Existenz der speziellen linearen Gruppe ergiebt.*

Statt mit Hülfe der vorigen Methode den Satz zu beweisen: " Eine lineare

Differentialgleichung 4. Ordnung kann mit Hülfe einer Quadratur auf eine nichtlineare Differentialgleichung 3. Ordnung reduciert werden," wollen wir gleich den allgemeinen Satz: "Eine lineare Differentialgleichung n^{ter} Ordnung kann mit Hülfe einer Quadratur auf eine nicht lineare Differentialgleichung $(n-1)^{\text{ter}}$ Ordnung reduciert werden," beweisen.

Die Differentialgleichung n^{ter} Ordnung: Betrachten wir die allgemeine lineare Differentialgleichung n^{ter} Ordnung:

$$\frac{d^n y}{dx^n} + n\lambda_1 \frac{d^{n-1}y}{dx^{n-1}} + \frac{n(n-1)}{1.2}\lambda_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + \lambda_n y = 0. \tag{1}$$

Ihre Transformationsgruppe ist die allgemeine homogene lineare Gruppe mit n Variabeln und n^2 Parametern. Diese Gruppe enthält eine invariante Untergruppe mit $n^2 - 1$ Parametern, die spezielle lineare Gruppe. Diese letztere ist einfach.

Aus der obigen allgemeinen Theorie folgt, dass diese Differentialgleichung mittelst einer Quadratur auf eine nicht lineare Differentialgleichung $(n-1)^{\text{ter}}$ Ordnung reduciert werden kann.

Die Invariante der speciellen linearen Gruppe ist:

$$D = |y_1 y_2' \ldots y_n^{(n-1)}| \tag{2}$$

welche

$$D' + n\lambda_1 D = 0$$

genügt. Die gegebene Gleichung (1) sollte sich auf eine nicht lineare Differen tialgleichung $(n-1)^{\text{ter}}$ Ordnung reducieren. Sei

$$u = \frac{y_1'}{y_1} \tag{4}$$

so können wir ohne weiteres aufschreiben:

$$\begin{aligned}
y_1' &= uy_1, \\
y_1'' &= u'y_1 + u^2 y_1, \\
y_1''' &= u''y_1 + 3uu'y_1 + u^3 y_1, \\
y_1^{(\mathrm{IV})} &= u'''y_1 + 4uu''y_1 + 6u^2u'y_1 + 3u'^2 y_1 + u^4 y_1, \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

$y_1^{(n)}$ wird $u^{(n-1)}$ und niedrigere Ableitungen von u, u^n und y_1 enthalten, und wenn wir diese Werte in (1) einsetzen, so erhalten wir eine Gleichung von der Gestalt

$$u^{(n-1)} + p_1 u^{(n-2)} + \ldots + p_n = 0 \tag{5}$$

wo $p_1 p_2 \ldots p_n$ bekannte Funktionen der λ und u sind.

Integrieren wir (3), so erhalten wir

$$D = e^{-n\int \lambda_1 dx} \tag{3'}$$

und integrieren wir jetzt (5) so ist hiemit die Gleichung (1) integriert.

Führen wir dies genauer aus. Nehmen wir an, es seien $u_1, u_2, \ldots. u_n$, u, $n+1$ Integrale der Gleichung (5). Setzen wir ferner:

$$u_1 = \frac{y_1'}{y_1}, \quad u_2 = \frac{y_2'}{y_2} \ldots. u_n = \frac{y_n'}{y_n}$$

so ist

$$u = \frac{y_1' + y_2' + \ldots. + y_n'}{y_1 + y_2 + \ldots. + y_n}$$

und daher

$$(u - u_1)\, y_1 + (u - u_2)\, y_2 + \ldots. + (u - u_n)\, y_n = 0$$

und wenn man letztere Gleichung differentiert, folgt:

$$[u_1 - u_1' + u_1(u - u_1)]\, y_1 + \ldots. + [u' - u_n' + u_n(u - u_n)]\, y_n = 0.$$

Wenn wir im Ganzen $n-2$ mal differentieren, und jedes mal y_i' durch $u_i y_i$ ersetzen, so erhalten wir im Ganzen $n-1$ homogene Gleichungen, und wir können daher schreiben:

$$\frac{y_1}{\xi_1} = \frac{y_2}{\xi_2} = \ldots. = \frac{y_n}{\xi_n} = \rho$$

wo $\xi_1 \ldots. \xi_n$ bekannte Funktionen von $u_1, \ldots. u_n, u$ und ihren Ableitungen sind. Wir haben daher:

$$D = \begin{vmatrix} y_1, & u_1 y_1, & (u_1' + u_1^2)\, y_1, & \ldots. & (u_1^{(n-1)} + nuu_1^{(n-2)} + \ldots.)\, y_1 \\ y_2, & u_2 y_2, & (u_2' + u_2^2)\, y_2, & \ldots. & (u_2^{(n-1)} + nuu_2^{(n-2)} + \ldots.)\, y_2 \\ \vdots & \vdots & \vdots & & \\ y_n, & u_n y_n, & (u_n' + u_n^2)\, y_n, & \ldots. & (u_n^{(n-1)} + nuu^{(n-2)} + \ldots.)\, y_n \end{vmatrix}$$

folglich

$$D = y_1 y_2 \ldots. y_n \eta$$

wo η eine bekannte Funktion der u und ihrer ersten $n-1$ Ableitungen ist.

Folglich ist:

$$D = \rho^n \xi_1 \xi_2 \ldots \xi_n \eta$$

und daher endlich:

$$y_1 = \rho\xi_1, \quad y_2 = \rho\xi_2, \ldots. y_n = \rho\xi_n, \quad \rho = D^{\frac{1}{n}} (\xi_1 \ldots. \xi_n \eta)^{-\frac{1}{n}}.$$

Das Auftreten der Wurzel rührt davon her, dass die Integration von (2) und (5) die endliche Transformationsgruppe der Gleichung auf die grösste gemeinschaftliche Untergruppe von D und $\frac{y_1'}{y_1} \dots \frac{y_n'}{y_n}$ reduciert, d. h. auf die nicht continuierliche Gruppe, die durch die n Transformationen

$$\bar{y}_1 = \varepsilon y_1, \quad \bar{y}_2 = \varepsilon y_2, \dots \bar{y}_n = \varepsilon y_n$$

wo $\varepsilon^n - 1 = 0$ bestimmt ist.

Soll die Transformationsgruppe von (1) die spezielle lineare Gruppe sein, so ist D rational bekannt ohne Quadratur, d. h. $n\lambda_1$ oder λ_1 selbst ist die logarithmische Ableitung einer rationalen Funktion (da $e^{-n\int \lambda_1 dx}$ rational ist) und wir brauchen bloss (5) zu integrieren.

Wenn in der Gleichung bereits das zweite Glied $\frac{d^{n-1}y}{dx^{n-1}}$ fehlt, so können wir schreiben

$$D = \begin{vmatrix} y_1, & y_1', & \dots & y_1^{(n-1)} \\ \vdots & & & \\ y_n, & y_n', & \dots & y_n^{(n-1)} \end{vmatrix} = 1.$$

Die Differentialgleichung 4ter *Ordnung:* Speciell für die Differentialgleichung 4ter Ordnung, wo die constanten Factoren der Coefficienten die Binomialcoefficienten sind, wird die Gleichung (5):

$$u''' + 3(u + 4\lambda_1)u'' + 6(u^2 + 2\lambda_1 u + \lambda_2)u' + 3u'^2$$
$$+ u^4 + 4\lambda_1 u^3 + 6\lambda_2 u^2 + 4\lambda_3 u + \lambda_4 = 0. \qquad (5')$$

Wenn wir diese letztere Gleichung in die zwei folgenden Gleichungen spalten

$$u''' + 3(u + 4\lambda_1)u'' + 6(u^2 + 2\lambda_1 u + \lambda_2)u' + 3u'^2 = 0, \qquad (6)$$
$$u^4 + 4\lambda_1 u^3 + 6\lambda_2 u^2 + 4\lambda_3 u + \lambda_4 = 0 \qquad (6')$$

so wird im allgemeinen ein Integral u_1 von (5′) nicht zugleich (6) und (6′) befriedigen; aber umgekehrt wird wenn u_1 ein Integral von (6) und eine Lösung von (6′) ist, u_1 auch (5′) genügen, und folglich wird

$$y = e^{\int u_1 dx}$$

ein Integral der gegebenen Differentialgleichung (1) sein.

Also gilt der Satz:

Wenn eine Wurzel der algebraischen Gleichung (6') *zugleich ein Integral der nicht linearen Differentialgleichung* 3^{ter} *Ordnung* (6) *ist, so ist sie die logarithmische Ableitung eines Integrales der allgemeinen linearen Differentialgleichung* (1).

So haben z. B. die nicht lineare Differentialgleichung 3^{ter} Ordnung.

$$u''' + 3(u - 2x)u'' + 6\left(u^2 - xu - \tfrac{1}{2}\right)u' + 3u'^2 = 8$$

und die algebraische Gleichung 4^{ten} Grades

$$u^4 - 2xu^3 - 3u^2 + 4X(u - x) + x^4 + 3x^2 = 0$$

(wo X eine beliebige Funktion von x ist), die gemeinsame Lösung $u_1 = x$, und daher hat die Differentialgleichung

$$\frac{d^4y}{dx^4} - 2x\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4X\frac{dy}{dx} - (4xX - x^4 - 3x^2)y = 0, \tag{1'}$$

$y_1 = ce^{\frac{x^2}{2}}$ zu einem Integral, wie sich leicht verificieren lässt.

Allgemeiner: wenn zwei Integrale u_1 und u_2 von (6) bekannt sind, so dienen sie dazu, λ_1 und λ_2 in (6) und weiterhin λ_3 und λ_4 in (6') zu bestimmen. Daher ist die lineare Differentialgleichung, von welcher $y_1 = e^{\int u_1 dx}$ und $y_2 = e^{\int u_2 dx}$ zwei Integrale sind, bestimmt. Die Trennung von (5') in (6) und (6') ist daher gleichbedeutend mit der Aufstellung zweier Bedingungen.

So wird z. B. die Differentialgleichung

$$u''' + 3\left(u - \frac{6x^2+1}{x}\right)u'' + 6\left(u^2 - \frac{6x^2+1}{x}u + 2x^2\right)u' + 3u'^2 = 0,$$

befriedigt durch $u_1 = x$ und $u_2 = 2x$; diese beiden Werte sind aber auch Wurzeln der Gleichung

$$u^4 - \frac{6x^2+1}{x}u^3 + 12x^2u^2 - (9x^3 - 7x)u + 2x^4 - 6x^2 = 0$$

und daher sind, wie man leicht verificiert, $y_1 = c_1e^{\frac{x^2}{2}}$ und $y_2 = c_2e^{x^2}$ zwei Integrale von

$$y^{\mathrm{IV}} - \frac{6x^2+1}{x}y''' + 12x^2y'' - (9x^3 - 7x)y' + (2x^4 - 6x^2)y = 0. \tag{1''}$$

Man wolle hiebei beachten, dass die coefficienten in der gegeben Differentialgleichung (1) und in der algebraischen Gleichung (6') genau dieselben sind.

Die Gleichung (5′) kann auch zerlegt werden in eine lineare Differentialgleichung 3^{ter} Ordnung:

$$u''' + 12\lambda_1 u'' + 6\lambda_2 u' + 4\lambda_3 u = 0 \tag{7}$$

und in eine nicht lineare Differentialgleichung 2^{ter} Ordnung:

$$3uu'' + 6(u + 2\lambda_1)uu' + 3u'^2 + u^4 + 4\lambda_1 u^3 + 6\lambda_2 u^2 + \lambda_4 = 0. \tag{8}$$

Wenn u_1 gleichzeitig ein Integral von (7) und (8) ist, so ist $e^{\int u_1 dx}$ ein Integral von (1).

Also gilt der Satz:

Wenn die logarithmische Ableitung des allgemeinsten Integrals der linearen Differentialgleichung 4^{ter} Ordnung (1) *der nicht linearen Differentialgleichung*

$$3uu'' + 6(u + 2\lambda_1)uu' + 3u'^2 + u^4 + 4\lambda_1 u^3 + 6\lambda_2 u^2 + \lambda_4 = 0 \tag{8}$$

genügt, so genügt sie auch der linearen Differentialgleichung 3^{ter} Ordnung:

$$u''' + 12\lambda_1 u'' + 6\lambda_2 u' + 4\lambda_3 u = 0, \tag{7}$$

und wenn die logarithmische Ableitung eines Integrals von (1) *der Gleichung* (7) *genügt, so genügt sie auch der Gleichung* (8).

§9.—*Zweite Reduction—Kubische Relation zwischen den Integralen.*

Wir werden nun an Stelle von x in §3 setzen y, und annehmen, dass die Transformationsgruppe unserer linearen Differentialgleichung 4^{ter} Ordnung den Typus IX hat. Die Invariante 3^{ter} Ordnung ist gegeben als Schnitt der zwei Oberflächen:

$$\begin{aligned}(1)\quad & y_2y_4 - y_1^2 = 0 \\ (2)\quad & y_3y_4^2 - y_1^3 = 0\end{aligned} \quad \text{oder als (3)} \quad \begin{aligned} & y_2y_4 - y_1^2 = 0, \\ & y_2^3 - y_3^2y_4 = 0.\end{aligned}$$

Das Problem kann jetzt, wie folgt, formuliert werden:

Eine lineare Differentialgleichung 4^{ter} Ordnung zu integrieren, wenn man weiss, dass eine kubische Relation von der Form [3] *zwischen ihren Integralen existiert, d, h. dass die Gruppe unserer Gleichung die Gruppe IX ist.*

Da die fragliche Gruppe nicht integrabel ist, so ist es unmöglich, die Gleichung durch Quadraturen zu integrieren.

Da die Gruppe IX die Untergruppe X mit einem Parameter weniger enthält, so folgt aus der allgemeinen Theorie, dass die Integration unserer Gleichung von der einer Differentialgleichung 1ter Ordnung abhängt. In der Tat werden wir sehen, dass es eine Riccati'sche Gleichung ist; aber wir wollen zuerst die Differentialgleichung 2ter Ordnung betrachten, deren Resolvente sie ist.

Die Curve (3) kann mit Hülfe zweier Parameter, wie folgt, dargestellt werden:

$$y_1 = u_1^2 u_2, \quad y_2 = u_1 u_2^2, \quad y_3 = u_2^3, \quad y_4 = u_1^3, \tag{4}$$

da ja (1) und (2) geschrieben werden können:

$$\frac{y_1}{y_2} = \frac{y_4}{y_1} = \frac{y_2}{y_3} = \frac{u_1}{u_2}.$$

Nehmen wir an, dass die Differentialgleichung 2ter Ordnung, für welche u_1 und u_2 zwei unabhängige Integrale sind, die folgende sei:

$$u'' + Ku = 0. \tag{5}$$

Um die Differentialgleichung zu bilden, von welcher $y_1 \ldots\ldots y_4$ abhängen, setzen wir:

$$y = u^3 \tag{6}$$

und nötigen Falls $y_i = u_i^3$ $(i = 1 \ldots\ldots 4)$.

Differenzieren wir (6) vier mal, indem wir mit Hülfe von (5) die Ableitungen von u eliminieren, die höher als die erste sind, so erhalten wir:

$$\begin{array}{lllll}
y^{(IV)} & = -\,60Kuu'^2 & -\,30K'u^2u' & +\,(21K^2 - 3K'')\,u^3, & \\
y''' & = & -\,21Ku^2u' & -\,3Ku^3 + 6u'^3, & \\
y'' & = \quad 6uu'^2 & & -\,3Ku^3, & \\
y' & = & 3u^2u' & & \\
y & = & & u^3 &
\end{array}$$

woraus dann folgt:

$$y^{(IV)} + 10Ky'' + 10K'y' + 3(3K^2 - 3K'')\,y = 0.$$

Und man sieht leicht, dass $u_1^2 u_2$ und $u_1 u_2^2$ ebenso zwei Integrale von (7) sind, wie u_1^3 und u_2^3.

Nun können wir statt der Gleichung (1), zufolge der Substitution $z = ye^{\int \lambda_1 dx}$ screiben:

$$\frac{d^4z}{dx^4} + 6p_2\frac{d^2z}{dx^2} + 4p_3\frac{dz}{dx} + p_4 z = 0, \tag{a}$$

wo
$$\left.\begin{aligned} p_2 &= \lambda_2 - \lambda_1' - \lambda_1^2, \\ p_2 &= \lambda_2 - 3\lambda_1\lambda_2 + 2\lambda_1^3 - \lambda_1'', \\ p_4 &= \lambda_4 - 4\lambda_1\lambda_3 + 6\lambda_1^2\lambda_2 - 3\lambda_1^4 - 6\lambda_2\lambda_1' + 6\lambda_1^2\lambda_1' + 3\lambda_1^2 - \lambda_1''', \end{aligned}\right\} \tag{8}$$

adjungieren wir also $e^{\int \lambda_1 dx}$ unserem Rationalitätsbereich, so kann man die Differentialgleichung (a) als die allgemeine lineare Differentialgleichung 4ter Ordnung ansehen.

Vergleichen wir nun die Gleichung (a) mit (7), so sehen wir, dass:

$$K = \tfrac{3}{5}\,p_2, \quad K' = \tfrac{2}{5}\,p_3, \quad 3\,(3K^2 - K'') = p_4 \tag{9}$$

und wenn wir K eliminieren, so erhalten wir

$$3p_2 = 2p_3, \quad p_4 - \quad p_3' - (\tfrac{9}{5})^2 p_2^2 = 0 \tag{10}$$

oder, durch die Coefficienten der gegebenen Gleichung ausgedrückt:

$$\left.\begin{aligned} &\lambda_1'' - 2\lambda_1\lambda_2' - 4\lambda_1^3 + 6\lambda_1\lambda_2 + 3\lambda_2' - 2\lambda_3 = 0 \\ &\text{und} \\ &\lambda_4 - 4\lambda_1\lambda_3 + 6\lambda_1^2\lambda_2 - 3\lambda_1^4 - 6\lambda_2\lambda_1' + 6\lambda_1^2\lambda_1' + 3\lambda_1'^2 - \lambda_1''' \\ &\qquad - \frac{6d}{5dx}\,(\lambda_3 - 3\lambda_1\lambda_2 + 2\lambda_1^3 - \lambda_1'') - (\tfrac{9}{5})^2\,(\lambda_2 - \lambda_1' - \lambda_1^2)^2 = 0. \end{aligned}\right\} \tag{11}$$

Wenn die zwei Relationen (11) zwischen den Coefficienten der gegebenen Gleichung, oder die zwei Relationen (10) zwischen den zwei Coefficienten der Gleichung, in welcher das zweite Glied fehlt, existieren, so können vier Integrale geschrieben werden:

$$y_1 = u_1^2 u_2, \quad y_2 = u_1 u_2^2, \quad y_3 = u_2^3, \quad y_4 = u_1^3, \tag{4}$$

wo u_1 und u_2 Integrale der linearen Differentialgleichung 2ter Ordnung und $u'' + ku = 0$ sind, und wo k sich aus einer der drei folgenden Gleichungen bestimmt:

$$k = \quad p_2, \quad k' = {}_{5}\,p_3, \quad k'' - 3k^2 = -\frac{p_4}{3},$$

Diese letzten drei Gleichungen müssen zugleich bestehen können.

Die von unserer Theorie geforderte Differentialgleichung 1ter Ordnung ist die entsprechende Riccati'sche Gleichung.

$$v' + v^2 + k = 0. \tag{12}$$

Die Relationen (11) *stellen die Bedingungen dafür dar, dass die Differentialgleichung* (1) *die Gruppe IX als ihre Transformationsgruppe habe* (§1, *II*).

§10.—*Dritte Reduction—Eine quadratische Relation zwischen den Integralen.*

Die quadratische Relation $y_1^2 - y_2 y_4 = 0$ ist offenbar ein Specialfall von

$$y_1 y_2 - y_3 y_4 = 0. \tag{1}$$

Um die Reduction der Differentialgleichung welche sich daraus ergibt zu untersuchen, schreiben wir:

$$u'' = \alpha u, \tag{2}$$

$$z'' = \beta z, \tag{3}$$

$$y = uz. \tag{4}$$

Differenzieren wir (4) 4 mal und eliminieren wir mittelst der ersten zwei Gleichungen die Ableitungen von höherer Ordnung als der zweiten, so erhalten wir:

$$\left.\begin{aligned}
y &= uz,\\
y' &= u'z + uz',\\
y'' &= (\alpha + \beta)\, uz + 2u'z',\\
y''' &= (\alpha' + \beta')\, uz + (\alpha + 3\beta)\, u'z + (3\alpha + \beta)\, uz',\\
y^{\mathrm{IV}} &= (\alpha'' + \beta'' + \alpha^2 + 6\alpha\beta + \beta^2)\, uz + (2\alpha' + 4\beta')\, u'z\\
&\qquad + (4\alpha' + 2\beta')\, uz' + 4\,(\alpha + \beta)\, u'z'.
\end{aligned}\right\} \tag{5}$$

Nachdem man -1, uz, $u'z$, uz', $u'z'$ eliminiert, ergibt sich folgende Differentialgleichung 4[ter] Ordnung

$$\begin{aligned}
y^{\mathrm{IV}} - \frac{\alpha' - \beta'}{\alpha - \beta}\, y''' - 2\,(\alpha + \beta)\, y'' + \left[\frac{\alpha' - \beta'}{\alpha - \beta}(\alpha + 3\beta) - 2\,(\alpha' + 2\beta')\right] y'&\\
+ \left[\frac{(\alpha' - \beta')(\alpha' + \beta')}{\alpha - \beta} - (\alpha'' + \beta'') + (\alpha - \beta)^2\right] y = 0.&
\end{aligned} \tag{6}$$

Beim Vergleichen mit der Gleichung (1, §1) erhält man:

$$\left.\begin{aligned}
-4\lambda_1 &= \frac{\alpha' - \beta'}{\alpha - \beta}, \quad -3\lambda_2 = (\alpha + \beta),\\
4\lambda_3 &= \frac{\alpha' - \beta'}{\alpha - \beta}(\alpha + 3\beta) - 2\,(\alpha' + 2\beta'),\\
\lambda_4 &= \frac{\alpha' - \beta'}{\alpha - \beta}(\alpha' + \beta') - (\alpha'' + \beta') + (\alpha - \beta)^2.
\end{aligned}\right\} \tag{7}$$

Glücklicherweise lässt sich α leicht eliminieren und es entsteht:

$$\left.\begin{aligned} (2\beta + 3\lambda_2)^2 &= \lambda_4 - 3\lambda_2'' - 12\lambda_1\lambda_2', \\ \beta' + 4\lambda_1\beta &= 3\lambda_2' + 6\lambda_1\lambda_2 - 2\lambda^3. \end{aligned}\right\} \tag{8}$$

Wenn wir β aus diesen zwei Gleichungen eliminieren, finden wir die Bedingung, welche zwischen den Coefficienten unserer Differentialgleichung bestehen muss, wenn $y_1 y_2 - y_3 y_4 = 0$ wird, mit andern Worten: wir finden so die Bedingung, dass die Gruppe unserer Gleichung, die Gruppe VIII sein soll.

Wenn u_1, u_2, z_1, z_2 *die Integrale von* (2) *und* (3) *sind dann werden die Integrale unserer Differentialgleichung* 4^{ter} *Ordnung sein:*

$$y_1 = u_1 z_1, \quad y_2 = u_2 z_2, \quad y_3 = u_2 z_1, \quad y_4 = u_1 z_2,$$

welche offenbar der verlangten Beziehung $y_1 y_2 - y_3 y_4 = 0$ *genügen.*

SCHLUSSBEMERKUNG

Lie hat gezeigt, dass seine Theorie auf alle damals bekannten Methoden, Differentialgleichungen zu integrieren, ein Licht wirft, indem sie alle diese verschiedenen aussehenden Methoden unter einen Gesichtspunkt bringt. Aber wie wir bereits bemerkt haben, ist seine Theorie für die Differentialgleichung nicht so vollständig, wie die entsprechende Theorie für die algebraischen Gleichungen.

Unsere Methode besteht kurz darin, möglichst viele der Gruppen in R_4 zu bestimmen, und die resultierenden Reductionen zu untersuchen, indem man annimmt, dass jede Gruppe der Reihe nach die Gruppe unserer Differentialgleichung ist. Das Problem kommt daher natürlicher Weise darauf hinaus, alle Gruppen in R_4 zu bestimmen.

Wir haben aus einer gewissen Anzahl dieser Gruppen, und zwar meistens primitiven, die linearen homogenen Gleider herausgegriffen. Aber dies genügt keineswegs. Es ist nämlich begreiflich, dass einige der weggelassenen Glieder in die linear homogene Form transformiert werden können; und diese müssen daher, um unser Problem vollständig zu lösen, auch noch untersucht werden. Die linearen homogenen Glieder, welche wir betrachtet haben, bilden wieder eine Gruppe für sich.

Um zu entscheiden ob die Gruppen integrabel sind oder nicht, haben wir Engel's Theorieen (§2) und die Methode der derivierten Gruppen angewandt. Wir sahen, dass von den 14 betrachteten Gruppen, nur diejenige, welche die Cayley'sche Oberfläche $3x_1 x_2 x_4 - x_3 x_4^2 - 2x_1^3 = 0$ invariant lässt, integrabel ist.

Unsere Ergebnisse scheinen daher anzudeuten, dass, wenn auch einige der Gruppen integrabel sind, bei den meisten dies noch nicht zutreffen dürfte, und dass daher die entsprechenden Reductionen nur teilweise nicht vollständige sind. Dies zeigt, dass wir im allgemeinen unsere Differentialgleichung nicht lösen können durch Integration einer Anzahl linearen Differentialgleichungen 1^{ter} Ordnung, aber wir können das Problem der Integration reducieren auf die Integration

einer oder mehrer Differentialgleichungen, die von niedrigerer Ordnung, als der 4^{ter} sind.

Wie zu erwarten war, zeigte sich, dass alle Gruppen algebraisch sind, und wir sahen (§1, I; §2), dass dies die einzige Art ist, mit der wir uns zu beschäftigen haben.

Da die Gruppen in §3 entweder eine kubische oder eine quadratische Relation zwischen den Integralen invariant lassen, so ergibt sich die Aufgabe, die entsprechenden Reductionen zu bestimmen. In §9 haben wir ein Beispiel einer kubischen und in §10 zwei Beispiele einer quadratischen Relation zwischen den Integralen.

Es bleibt endlich noch die schwierigste Aufgabe (§1, IV) zu lösen, nämlich: "die Gruppe einer gegebenen Differentialgleichung zu bestimmen." Aber diese Aufgabe liegt zu weit asserhalb des Rahmens der vorliegenden Arbeit, und muss einer späteren Gelegenheit überlassen werden.*

Ich ergreife die Gelegenheit gerne, Herrn Professor Burkhardt meinen besondern Dank auszusprechen für seine Liebenswürdigkeit und seine anregenden Ratschläge.

ZÜRICH, *Aug.*, 1901.

* Siehe die Textnote, Seite 129.

INHALTSVERZEICHNIS.

III. KAPITEL.

The Logic of Relations, Logical Substitution Groups, and Cardinal Numbers.

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Preface.

In Section I, the theory of logical equations is generalized; any definite logical equation is proved to correspond to a definite class of relations [cf. ★1·1 and ★1·2], and to each relation of the class corresponds a solution of the equation. But from the relational point of view the theory is equally simple, whether the number of variables in the corresponding logical equation is finite or infinite. Accordingly, we obtain a theory of logical equations when the cardinal number of the variables has any infinite value. The solution of this general type of equation is found [cf. ★3·32 and ★4·04]. This is effected by the help of some important definitions [cf. ★2·0, and ★2·05, and ★2·22, and ★3·10, and ★3·20]. In ★5, the application to equations with a finite number of variables is considered.

In Section II, Cantor's theory of cardinals as developed in my paper on "Cardinal Numbers" in Vol. XXIV, p. 367 of this Journal, is applied; and after determining the cardinal numbers of various classes of relations in ★10, in ★11 the number of solutions of any logical equation is determined [cf. ★11·13 and ★11·25]. In ★12, these results are considered for the special case of a finite number of variables [cf. ★12·01, and ★12·02, and ★12·2], and some examples for one and two variables are appended. In ★13, the following problem is considered: i is a given class, a and b are given classes contained in i, required the number of classes x contained in i such that the cardinal number of the class $(a \frown x) \smile (b \frown \bar{x})$, where $\bar{x}$ is the part of i not x, is some given number α. This number is determined [cf. ★ 13·20 to ★13·24]. Thence

[cf. ★13·30], with the same suppositions, the sum of the following series is determined:

$$\sum^{x)i} 2^{\mu\{(a \frown x) \smile (b \frown \breve{x})\}}.$$

These two sections are written out in the notations of Peano and Russell, explained in the memoir on "Cardinal Numbers" (loc. cit.).

Section III considers the orders of the Logical Substitution Groups, considered in my memoir on "Symbolic Logic," in Vol. XXIII, p. 297, of this Journal; the order of the complete group is $24^{\mu i}$ (cf. ★20·1); the order of the identical group of a function with invariants s_1, s_2, s_3, s_4 is

$$24^{\mu(\bar{s}_1 \smile s_4)} \times 6^{\mu\{(s_1 \frown \bar{s}_2) \smile (s_3 \frown \bar{s}_4)\}} \times 4^{\mu(s_2 \frown \bar{s}_3)}.$$

Also the orders of other groups are determined.

Section IV deals with some properties of a certain simple type of substitutions.

My memoir on "Symbolic Logic" in this Journal, Part I in Vol. XXIII, p. 140, and Part II, Vol. XXIII, p. 297, is always cited as Symb. Log., Part I or Symb. Log., Part II; the memoir "On Cardinal Numbers" in Vol. XXIV of his Journal is cited as Card. Numb.

SECTION I.

★1 $i, h \,\varepsilon\, \mathrm{cls} \,.\, P \,\varepsilon\, \mathrm{rel} \,.\, \pi \supset i \,.\, \breve{\pi} \supset h \,.\supset .\because$

·1 $\mathrm{equ}\,(i, h, P) = \mathrm{rel} \frown R \ni [\rho = i \,.\, \breve{\rho} \supset h \,.\, \breve{R}P \supset 0']$. Df.

Note: $\breve{R}P \supset 0' .=. R\breve{P} \supset 0'$ [cf. Card. Numb., Section II, 2·13]. Here "equ" is contracted from "equation." The connection between this definition and the ordinary theory of logical equations is most easily seen from the next proposition,

·2 $\mathrm{equ}\,(i, h, P) = \mathrm{rel} \frown R \ni [\rho = i \,.\, \breve{\rho} \supset h : k \,\varepsilon\, h \,.\supset_k .\, \pi k \frown \rho k = \Lambda]$,
[★1·1 .=. Prop.]

Note: To establish the connection between these propositions and the theory of logical equations, consider h as the class of indices not necessary finite or denumerable in number: i is the class called the universe, and all the classes appearing in the equation as factors or as summands are contained in i; P is the relation determining the known

coefficients of the various terms, thus πk may be written a_k, where a_k is a known class contained in i and corresponding to the index k; since $\breve{\pi} \supset h$ and is not necessarily equal to h, it may happen that $a_k = \Lambda$; R is the relation determining the unknowns of the equation, thus: let ρk be written x_k, where x_k is a class contained in i and corresponding to the index k: then ★1·1 and the general hypothesis assert that for every product of the type $a_k \cap x_k$ we have

$$a_k \cap x_k = \Lambda,$$

and that the logical sum of all classes of the type x_k is equal to i. For example, if the number of indices is two, so that $\mu h = 2$ and these indices are 1 and 2, then

$$a_1 \cap x_1 \cup a_2 \cap x_2 = \Lambda, \quad x_1 \cup x_2 = i.$$

In logical equations, as ordinarily considered, we should also have $x_1 \cap x_2 = \Lambda$, so that $x_2 = \bar{x}_1$ (putting $\bar{x}_1$ for $i \sim x_1$), and the equation becomes

$$a_1 \cap x_1 \cup a_2 \cap \bar{x}_1 = \Lambda.$$

This further specialization of the general idea will be considered later; but meanwhile we shall prove a series of propositions which belong equally to the more general conception here defined.

★2 $i, h \,\varepsilon\, \text{cls} \,.\, P \,\varepsilon\, \text{rel} \,.\, \pi \supset i \,.\, \breve{\pi} \supset h \,.\, \supset \,.\!\cdot\,.$

·0 $b \supset h \,.\, \supset \,.\, (b, \text{div}\, P) = i \cap x \ni (\breve{\pi} x = h \sim b)$. Df.

·01 $b \supset h \,.\, \exists\, h \sim b \sim \breve{\pi} \,.\, \supset \,.\, (b, \text{div}\, P) = \Lambda,$

$[\text{Hp} \,.\, \supset \,.\, \sim \exists\, i \cap x \ni (h \sim b = \breve{\pi} x) \,.\, \supset \,.\, \text{Prop}].$

·02 $\breve{\pi} = h \,.\, \supset \,.\, (\Lambda, \text{div}\, P) = x \ni (\breve{\pi} x = h).$

·03 $\breve{\pi} \sim = h \,.\, \supset \,.\, (\Lambda, \text{div}\, P) = \Lambda,$

$[\text{Hp} \,.\, \supset \,.\, \sim \exists\, i \cap x \ni (\breve{\pi} x = h) \,.\, \supset \,.\, \text{Prop}].$

·04 $(h, \text{div}\, P) = i \sim \pi,$

$[(h, \text{div}\, P) = i \cap x \ni (\breve{\pi} x = \Lambda) \,.\, \supset \,.\, \text{Prop}].$

·05 $(Nc, \text{div}\, P) = y \ni [\exists\, \text{cls}'h \cap b \ni \{y = (b, \text{div}\, P)\}]$. Df.

Note: "div" is contracted from "divisional": the importance of a similar conception in relation to logical equations containing a finite number of variables was exemplified by W. E. Johnson in a paper read

before the International Congress of Philosophy, Paris, 1900, in the section dealing with "Logique et Histoire des Sciences" (published by Armand Colin, Paris). The definitions, ★2·0 and ★2·05, have essential reference to i, h which are given in the general hypothesis; in other connections it might be necessary to express these classes and to write $(b, \mathrm{div}_h^i\, P)$ for $(b, \mathrm{div}\, P)$ and $(Nc, \mathrm{div}_h^i\, P)$ for $(Nc, \mathrm{div}\, P)$.

·10 $P_{\breve{\pi}} = \iota\, \mathrm{rel} \cap P_1 \ni (\pi_1 = i : x \varepsilon i . x P_1 y . \supset . y = \breve{\pi} x)$. Df.

·11 $\breve{\pi}_{\breve{\pi}} \supset \mathrm{cls}' h \cdot \cup ‘\breve{\pi}_{\breve{\pi}} = \breve{\pi}$.

·12 $P_{\breve{\pi}} \varepsilon Nc \rightarrow 1$.

·13 $b \supset h . \supset . (b, \mathrm{div}\, P) = \pi_{\breve{\pi}}(h \sim b)$.

·20 $(Nc, \mathrm{div}\, P) \varepsilon \mathrm{cls}^2\, \mathrm{excl}$,

$[P_{\breve{\pi}} \varepsilon Nc \rightarrow 1 . \supset : x P_{\breve{\pi}} y . x' P_{\breve{\pi}} y' . y o' y' . \supset . x o' x' : \supset . \mathrm{Prop}]$.

·21 $\cup ‘(Nc, \mathrm{div}\, P) = i$,

$[x \varepsilon \pi . \subset : b = h \sim \breve{\pi} x . \supset . x \varepsilon (b, \mathrm{div}\, P) : \supset . x \varepsilon^2 (Nc, \mathrm{div}\, P)$, (1)

★ $2·04 . \supset : x \varepsilon i \sim \pi . \supset . x \varepsilon (h, \mathrm{div}\, P) . \supset . x \varepsilon^2 (Nc, \mathrm{div}\, P)$, (2)

$(1) . (2) . \supset . \mathrm{Prop}]$.

Note: The importance of the class $(Nc, \mathrm{div}\, P)$ depends upon ★2·20 and ★2·21.

·22 $\beta \varepsilon Nc . \supset . (\beta, \mathrm{div}\, P) = x \ni [\exists\, \mathrm{cls}' h \cap b \ni \{b \varepsilon \beta . x = (b, \mathrm{div}\, P)\}]$. Df.

★3 $i, h \varepsilon \mathrm{cls} . P \varepsilon \mathrm{rel} . \pi \supset i . \breve{\pi} \supset h . \exists\, \mathrm{equ}\,(i, h, P) . R \varepsilon\, \mathrm{equ}\,(i, h, P) . \supset .\therefore$

·01 $b \supset h . a \varepsilon (b, \mathrm{div}\, P) . a R k . \supset . k \varepsilon b$,

$[\mathrm{Hp} . \supset : a \varepsilon i . h \sim b = \breve{\pi} a . \supset : x \varepsilon h \sim b . \supset_x . a P x$, (1)

$(1) . x \varepsilon h \sim b . a R k . \breve{R} P \supset o' . \supset . x o' k$, (2)

$\mathrm{Hp} . (2) . \supset . \mathrm{Prop}]$.

·02 $(\Lambda, \mathrm{div}\, P) = \Lambda$,

$[★ 3·01 . \rho = i . \supset : b \supset h . \exists\, (b, \mathrm{div}\, P) . \supset . \exists\, b : \supset . \mathrm{Prop}]$.

·10 $b \supset h . \exists\, (b, \mathrm{div}\, P) . \supset . (b, \mathrm{rel}\, P) = \mathrm{rel} \cap S \ni [\sigma = (b, \mathrm{div}\, P) . \breve{\sigma} \supset b]$. Df.

Note: $(b, \mathrm{rel}\, P)$, like $(b, \mathrm{div}\, P)$, refers essentially to i and h which are given in the general hypothesis. If it were necessary to render these classes explicit in the notation, we could write $(b, \mathrm{rel}_h^i\, P)$ for $(b, \mathrm{rel}\, P)$.

·11 $\beta \varepsilon Nc . \supset . (\beta, \mathrm{rel}\, P) = x \ni [\exists\, \mathrm{cls}' h \cap b \ni \{b \varepsilon \beta . x = (b, \mathrm{rel}\, P)\}]$. Df.

·12 $(Nc, \mathrm{rel}\, P) = x \ni [\exists\, \mathrm{cls}' h \cap b \ni \{x = (b, \mathrm{rel}\, P)\}]$. Df.

·13 $S \varepsilon^2 (Nc, \text{rel } P) . = . S \varepsilon \text{rel} . \exists \text{ cls' } h \frown b \ni [\sigma = (b, \text{div } P) . \breve{\sigma} \supset b]$.

·14 $S, S' \varepsilon^2 (Nc, \text{rel } P) . S o' S' . \supset : \sigma \frown \sigma' = \Lambda . \smile . \sigma = \sigma'$,
$[\star 2{\cdot}20 . \star 3{\cdot}13 . \supset . \text{Prop}]$.

·20 $\{\text{rel } (Nc, \text{rel } P)^{\times}\} = \text{rel} \quad S \ni [\exists (Nc, \text{rel } P)^{\times} \frown M \ni (S = \smile {}^{\backprime}M)]$. Df.

Note: For an explanation of this use of the symbol $\times$, cf. Card. Numb. $\star 6{\cdot}0$.

·21 $S \varepsilon \{\text{rel} (Nc, \text{rel } P)^{\times}\} . \supset . \sigma = \smile {}^{\backprime}(Nc, \text{div } P) = i$.

·22 $S \varepsilon \{\text{rel} (Nc, \text{rel } P)^{\times}\} . b \supset h . \exists \ (b, \text{div } P) . \supset .$
$S_b = \imath (b, \text{rel } P) \frown T \ni [x \varepsilon b . \supset : x \, Ty . = . x \, Sy]$. Df.

·23 Hp $\star 3{\cdot}22 . \supset . \sigma_b = (b, \text{div } P) . \breve{\sigma}_b \supset b$.

·24 $S \varepsilon \{\text{rel} (Nc, \text{rel } P)^{\times}\} . b \supset h . b' \supset h . b o' b' . \supset . \sigma_b \frown \sigma_{b'} = \Lambda$.

·25 $S \varepsilon \{\text{rel} (Nc, \text{rel } P)^{\times}\} . \supset . \breve{\sigma} \supset h$.

·26 $S \varepsilon \{\text{rel} (Nc, \text{rel } P)^{\times}\} . \supset . \breve{S} P \supset o'$,
$[x \varepsilon \breve{\sigma} . \supset . \exists \text{ cls' } h \frown b \ni \{x \varepsilon b . \exists (b, \text{div } P)\}$, (1)
$(1) . x \breve{S} z . \supset . z \varepsilon (b, \text{div } P)$, (2)
$z \varepsilon (b, \text{div } P) . z P y . \supset . y \varepsilon h \sim b$, (3)
$(1) . (2) . (3) . \supset : x \breve{S} P y . \supset . \exists \text{ cls' } h \frown b \ni \{x \varepsilon b . y \varepsilon h \sim b\} : \supset . \text{Prop}]$.

·30 $\{\text{rel} (Nc, \text{rel } P)^{\times}\} \supset \text{equ} (i, h, P)$,
$[\star 3{\cdot}21 . \star 3{\cdot}25 . \star 3{\cdot}26 . \supset . \text{Prop}]$.

·31 $\text{equ} (i, h, P) \supset \{\text{rel} (Nc, \text{r l } P)^{\times}\}$,
$[R \varepsilon \text{equ} (i, h, P) . \supset .\because$
$\star 2{\cdot}20 . \star 2{\cdot}21 . a \varepsilon i . \supset . \text{cls' } h \frown b \ni \{a \varepsilon (b, \text{div } P)\} \varepsilon 1$, (1)
$(1) . \star 3{\cdot}01 . a \varepsilon i . b \varepsilon \imath \text{ cls' } h \frown b \ni \{a \varepsilon (b, \text{div } P)\} . a R k . \supset . k \varepsilon b$, (2)
$(2) . b \supset h . \exists (b, \text{div } P) . \supset . R_b \ni [\rho_b = (b, \text{div } P) . \breve{\rho}_b \supset b .\because$
$a \varepsilon (b, \text{div } P) . \supset_a : a R k . = . a R_b k] \varepsilon 1 \frown \text{cls' } (b, \text{rel } P)$, (3)
$\rho = i . \star 2{\cdot}21 . (4) . \supset . R \varepsilon \{\text{rel } (Nc, \text{rel } P)^{\times}\} . \supset . \text{Prop}]$.

·32 $\text{equ} (i, h, P) = \{\text{rel} (Nc, \text{rel } P)^{\times}\}$,
$[\star 3{\cdot}30 . \star 3{\cdot}31 . \supset . \text{Prop}]$.

Note: $\star 3{\cdot}32$ gives the general solution for the class of relations indicated by equ (i, h, P), in the sense that $\{\text{rel} (Nc, \text{rel } P)^{\times}\}$, which has been proved to be the same class, is defined by indicating a method for the construction of any member of the class, whereas the definition of equ (i, h, P) simply indicates the general property of any member

of the class: we have here an example of two different class-concepts with the same extension.

We now proceed to specialize these ideas in the direction of ordinary logical equations.

★4 $i, h \, \varepsilon \, \mathrm{cls} \, . \, P \, \varepsilon \, \mathrm{rel} \, . \, \pi \supset i \, . \, \breve{\pi} \supset h \, . \, \exists \; \mathrm{equ}\,(i, h, P) \, . \supset .\!\cdot\!.$

·0 $Nc \rightarrow 1 \cap \mathrm{equ}\,(i, h, P) = Nc \rightarrow 1 \cap R \ni \{\rho = i \, . \, \breve{\rho} \supset h \, . \, \breve{R} \, P \supset o'\}.$

·01 $b \supset h \, . \, \exists \, (b, \mathrm{div}\, P) \, . \supset . \, (b, Nc \rightarrow 1, P) = Nc \rightarrow 1 \cap (b, \mathrm{rel}\, P).$ Df.

·02 $(Nc, Nc \rightarrow 1, P) = x \ni [\exists \; \mathrm{cls}'h \cap b \ni \{x = (b, Nc \rightarrow 1, P)\}].$ Df.

·03 $\{\mathrm{rel}\,(Nc, Nc \rightarrow 1, P)^{\times}\}$

$= \mathrm{rel} \cap S \ni [\exists \, (Nc, Nc \rightarrow 1, P)^{\times} \cap M \ni \{S = \cap \,{}^{\backprime}M\}].$ Df.

·04 $Nc \rightarrow 1 \cap \mathrm{equ}\,(i, h, P) = \{\mathrm{rel}\,(Nc, Nc \rightarrow 1, P)^{\times}\},$

[★3·32 . ⊃ . Prop].

Note: With the notation of the note on ★1·2, we have, if $R \, \varepsilon \, Nc \rightarrow 1 \cap \mathrm{equ}\,(\iota, h, P),$

$$a_k \cap x_k = \Lambda,$$

and the logical sum of all the classes of the type x_k is equal to i, and the logical product of any two different classes of the type x_k, say x_k and $x_{k'}$, where k is different from k', is nu l, that is,

$$k, k' \, \varepsilon \, h \, . \, k \, o' \, k' \, . \supset . \, x_k \cap x_{k'} = \Lambda.$$

Thus the class of classes of the type x_k is exhaustive of i and the classes are mutually exclusive. For instance, if the number of indices is 2, so that $h \, \varepsilon \, 2$, and if $\bar{x}_1$ is put for $i \sim x_1$, then $x_1 \cup x_2 = i$ and $x_1 \cap x_2 = \Lambda$; hence $x = \bar{x}_1$, and the equation becomes

$$a_1 \cap x_1 \cup a_2 \cap \bar{x}_1 = \Lambda.$$

The general relation of the above theorems to logical equations with a finite number of unknowns is considered in the next set of propositions ★5·0 to ★5·11; and equations with a finite number of variables are again considered in set ★12

We shall use the following notation wherever the symbol i represents a class

$x \supset i \, . \supset_x . \, \bar{x} = i \sim x.$

★5 $i \, \varepsilon \, \mathrm{cls} : x \supset i \, . \supset_x . \, \bar{x} = i \sim x : \nu \, \varepsilon \, Nc \; \mathrm{fin} \, . \, h = Nc \cap \beta \ni (o < \beta \leq \nu):$

$P \, \varepsilon \, \mathrm{rel} \, . \, \pi \supset i \, . \, \breve{\pi} \supset h \, . \, \exists \; \mathrm{equ}\,(i, h, P) : \beta \, \varepsilon \, h \, . \supset . \, a_\beta = \pi \beta :$

$R \, \varepsilon \, \mathrm{equ}\,(i, h, P) \, . \, \beta \, \varepsilon \, h \, . \supset . \, x_\beta = \rho \beta : \supset .\!\cdot\!.$

•0 $\lambda \varepsilon Nc . 0 < \lambda \leqq \nu . b = \iota 1 \cup \iota 2 \cup \iota 3 \cup \ldots . \iota \lambda . \supset .$
$(b, \text{div } P) = \bar{a}_1 \frown \bar{a}_2 \frown \ldots . \frown \bar{a}_\lambda \frown a_{\lambda+1} \frown a_{\lambda+2} \frown \ldots . \frown a_\nu$
$[\bigstar 2{\cdot}0 . \supset . \text{Prop}].$

•01 $(\Lambda, \text{div } P) = a_1 \frown a_2 \frown \ldots . \frown a_\nu = \Lambda,$
$[\bigstar 2{\cdot}0 . \bigstar 3{\cdot}02 . \text{Hp} . \supset . \text{Prop}].$

.02 $(h, \text{div } P) = \bar{a}_1 \frown \bar{a}_2 \frown \ldots . \frown \bar{a}_\nu.$

•03 $\beta \varepsilon h . A = Z \ni [\exists\, h \frown \lambda \ni (Z = a_\lambda)] .$
$D^A_\beta = y \ni [\exists\, C^A_\beta \frown u \ni (y = \frown \text{‘}u)] . \supset . S_\beta = \cup \text{‘}D^A_\beta.$ Df

Note: $S_1, S_2, \ldots S_\nu$ are the symmetric functions of $a_1, a_2, \ldots, a.$ as defined in "Symb. Logic," Part I, §2; thus, $S_1 = a_1 \cup a_2 \cup \ldots \cup a_\nu$ and $S_\nu = a_1 \frown a_2 \frown \ldots \frown a_\nu$.

•04 $S_0 = i.$ Df.

Note: This definition is convenient to preserve the generality of certain formulæ.

•05 $\beta \varepsilon Nc . \beta \leqq \nu . \supset . \cup \text{‘}(\beta, \text{div } P) = S_{\nu-\beta} \frown \bar{S}_{\nu-\beta+1},$
$[\bigstar 2{\cdot}22 . \bigstar 5{\cdot}0 . \bigstar 5{\cdot}03 . \supset . \text{Prop}].$

•06 $S_\nu = \Lambda,$ $[\bigstar 5{\cdot}01 . = . \text{Prop}].$

•10 $R \varepsilon \text{ equ } (i, h, P) . \supset : (a_1 \frown x_1) \cup (a_2 \frown x_2) \cup \ldots . \cup (a_\nu \frown x_\nu) = \Lambda .$
$x_1 \cup x_2 \cup \ldots . \cup x_\nu = i,$
$[\text{Hp} (\bigstar 5) . \supset . \text{Prop}].$

•11 $R \varepsilon Nc \rightarrow 1 \frown \text{equ} (i, h, P) . \supset .\cdot. \bigstar 5{\cdot}10 : \lambda, \lambda' \varepsilon h .$
$\lambda o' \lambda' . \supset . x_\lambda \frown x_{\lambda'} = \Lambda.$

Note: Comparing this with the ordinary type of logical equation, for instance, in two variables,

$$(a \frown x \frown y) \cup (b \frown x \frown \bar{y}) \cup (c \frown \bar{x} \frown y) \cup (d \frown \bar{x} \frown \bar{y}) = \Lambda,$$

we see that $x_1 = x \frown y$, $x_2 = x \frown \bar{y}$, $x_3 = \bar{x} \frown y$, $x_4 = \bar{x} \frown \bar{y}$. Thus $x = x_1 \cup x_2$, $y = x_1 \cup x_3$. Also for the comparison to hold, ν must be a number of the type 2^δ, and then δ is the number of unknowns in the ordinary logical equation. But whatever ν may be, the equation

$$(a_1 \frown x_1) \cup (a_2 \frown x_2) \cup \ldots . \cup (a_\nu \frown x_\nu) = \Lambda,$$

where $x_1 \cup x_2 \cup \ldots . \cup x_\nu = i$ and $x_\lambda \frown x_{\lambda'} = \Lambda$, $(\lambda o' \lambda')$ can always be modified into an equation of the required type.

For, let $\delta \leqq \nu \leqq 2^\delta$, and let $a_{\nu+1}, a_{\nu+2}, \ldots\ldots a_{2^\delta}$ be each equal to i, so that

$$\nu < \lambda \leqq 2^\delta . \supset . a_\lambda = i,$$

then $$\nu < \lambda \leqq 2^\delta . x_\lambda \supset i . a_\lambda \frown x_\lambda = \Lambda . \supset . x_\lambda = \Lambda .$$

Hence by adding on to $a_1, \ldots\ldots a_\nu$ the $(2^\delta - \nu)$ terms $a_{\nu+1}, \ldots\ldots a_{2^\delta}$ (all equal to i), and to $x_1, \ldots\ldots x_\nu$ the $(2^\delta - \nu)$ terms $x_{\nu+1}, \ldots\ldots x_{2^\delta}$ (all equal to Λ), we obtain

$$(a_1 \frown x_1) \cup (a_2 \frown x_2) \cup \ldots\ldots \cup (a_{2^\delta} \frown x_{2^\delta}) = \Lambda,$$

where $x_1 \cup x_2 \cup \ldots\ldots \cup x_{2^\delta} = i$ and $x_\lambda \frown x_{\lambda'} = \Lambda$, $(\lambda o' \lambda')$, and $x_1, x_2, \ldots\ldots x_\nu$ can be any set of terms satisfying the unmodified equation and can be no other set. Hence there is no loss of generality in supposing that ν is always of the form 2^δ. The next set of propositions (★6) will deal with the generalization of this reasoning for the case when ν may be infinite.

★6 $i, h \,\epsilon\, \mathrm{cls} . \nu \,\epsilon\, Nc . h \,\epsilon\, \nu . R \,\epsilon\, Nc \to 1 . \rho = i . \breve{\rho} \supset h . \supset .\therefore$

·10 $\exists\, Nc \frown \delta \ni (\delta \leqq \nu \leqq 2^\delta)$.

·2 $P \,\epsilon\, \mathrm{rel} . \pi \supset i . \breve{\pi} \supset h . \exists\, \mathrm{equ}\,(i, h, P) . \delta \,\epsilon\, Nc . \delta \leqq \nu \leqq 2^\delta .$
$h' \,\epsilon\, \mathrm{cls} . h \frown h' = \Lambda . h \cup h' = 2^\delta .$
$P' = \imath\, \mathrm{rel} \frown P'' \ni [\pi'' \supset i . \breve{\pi}'' \supset h \cup h' .\therefore z \,\epsilon\, h . \supset : x P' z . = . x P z .\therefore$
$z \,\epsilon\, h' . x \,\epsilon\, i . \supset_{x, z} . x P' z] . \supset : \mathrm{equ}\,(i, h \cup h', P') = \mathrm{equ}\,(i, h, P)$.

Note: This proposition, of which the proof is easy, shows that there is no loss of generality in always assuming, when convenient, ν to be of the form 2^δ.

SECTION II.

The Cardinal Numbers of Various Classes.

★10 $u, v \,\epsilon\, \mathrm{cls} . u \frown v = \Lambda . \supset .\therefore$

.01	$(\supset u, \mathrm{rel}, \supset v) = \mathrm{rel} \frown R \ni (\rho \supset u . \breve{\rho} \supset v)$.	Df.
·11	$(u, \mathrm{rel}, \supset v) = \mathrm{rel} \frown R \ni (\rho = u . \breve{\rho} \supset v)$.	Df.
·12	$(\supset u, \mathrm{rel}, v) = \mathrm{rel} \frown R \ni (\rho \supset u . \breve{\rho} = v)$.	Df.
·13	$(u, \mathrm{rel}, v) = \mathrm{rel} \frown R \ni (\rho = u . \breve{\rho} = v)$.	Df.
·2	$(u ; v) = (x, y) \ni (x \,\epsilon\, u . y \,\epsilon\, v)$.	Df.

·30 $\mu(u;v) = \mu u \times \mu v$, [cf. Card. Numb. ★ 7·21].

·31 $\mu(\supset u, \text{rel}, \supset v) = \mu \text{ cls}'(u; v) = 2^{\mu u \times \mu v}$, [cf. Card. Numb. ★ 15·0].

·32 $\mu(u, \text{rel}, \supset v) = (2^{\mu v} - 1)^{\mu u}$,

$[x \,\varepsilon\, u . \supset . k_x = l \ni \{l \supset \iota x \cup \text{cls}' v \sim \iota\Lambda . x \,\varepsilon\, l . l \cap \text{cls}' v \,\varepsilon\, 1\} .\therefore$

$k = p \ni \{\exists\, u \cap x \ni (p = k_x)\} :\supset . k \,\varepsilon\, \mu u . k \supset \mu \text{ cls}' v \sim \iota\Lambda$, (1)

$m \,\varepsilon\, k^{\times} . \supset .\therefore x \,\varepsilon\, u . \supset_x . m \cap k_x \,\varepsilon\, 1 : m \supset u \cup \text{cls}' v \sim \iota\Lambda . u \supset m :: \supset .$

$\mu k^{\times} = \mu(u, \text{rel}, \supset v)$, (2)

(Card. Numb. ★ 12·1) . (1) . (2) . $\supset$. Prop].

·33 $\mu u + \mu v \,\varepsilon\, Nc$ infin $. \mu u > 1 . \mu v > 1 . \supset .$

$\mu(u, \text{rel}, \supset v) = 2^{\mu u \times \mu v} = \mu(\supset u, \text{rel}, v)$.

·40 $Nc \rightarrow 1 \cap (u, \text{rel}, \supset v) = v^u$, [cf. Card. Numb. ★ 14·0].

·41 $\mu\{Nc \rightarrow 1 \cap (u, \text{rel}, \supset v)\} = \mu v^{\mu u}$, [cf. Card. Numb. ★ 14·1].

·51 $\mu\{Nc \rightarrow 1 \cap (\supset u, \text{rel}, \supset v\} = (1 + \mu v)^{\mu u}$,

$[w \,\varepsilon\, C^u_\beta . \supset . \mu\{Nc \rightarrow 1 \cap (w, \text{rel}, \supset v)\} = (\mu v)^\beta$, (1)

(1) $. \supset . \mu\{Nc \rightarrow 1 \cap (\supset u, \text{rel}, \supset v)\} = \sum^{\beta \leqq \mu u} C^{\mu u}_\beta \times (\mu v)^\beta$, (2)

(Card. Numb. ★ 17·4) . (2) . $\supset$. Prop].

.52 $\mu\{1 \rightarrow Nc \cap (\supset u, \text{rel}, \supset v)\} = (1 + \mu u)^{\mu v}$.

·61 $\mu u > 1 . \mu v > 1 . \supset : (2^{\mu v - 1} - 1)^{\mu u - 1} \leqq \mu(u, \text{rel}, v) \leqq (2^{\mu v} - 1)^{\mu u}$.

$(2^{\mu u - 1} - 1)^{\mu v - 1} \leqq \mu(u, \text{rel}, v) \leqq (2^{\mu u} - 1)^{\mu v}$,

$[x \,\varepsilon\, u . y \,\varepsilon\, v . R \,\varepsilon\, (u \sim \iota x, \text{rel}, \supset v \sim \iota y) . R' \,\varepsilon$ rel $. \rho' = \iota x .$

$\breve{\rho}' = v \sim \breve{\rho}(u \sim \iota x) . \supset . \exists\, \breve{\rho}' . \supset . R \cup R' \,\varepsilon\, (u, \text{rel}, v)$, (1)

(1) $. x \,\varepsilon\, u . y \,\varepsilon\, v . \supset . \mu(u, \text{rel}, v) \geqq \mu(u \sim \iota x, \text{rel}, \supset v \sim \iota y)$, (2)

(2) . ★ 10·32 . $(u, \text{rel}, v) \supset (u, \text{rel}, \supset v) . \supset$. Prop].

·62 $\mu u > 1 . \mu v > 1 . \mu u + \mu v \,\varepsilon\, Nc$ infin $. \supset . \mu(u, \text{rel}, v) = 2^{\mu u \times \mu v}$,

[★ 10·61 . $\supset$. Prop].

★ 11 $i, h \,\varepsilon$ cls $. P \,\varepsilon$ rel $. \pi \supset i . \breve{\pi} \supset h . \exists$ equ $(i, h, P) . \supset .\therefore$

·0 μ equ $(i, h, P) = \mu\{\text{rel}(Nc, \text{rel } P)^{\times}\} = \mu\{Nc, \text{rel } P\}^{\times}$.

[★ 3·32 . ★ 3·20 . $\supset$. Prop].

·01 μ equ $(i, h, P) = \prod^{\beta \leqq \mu h} \mu(\beta \text{ rel } P)^{\times}$,

[(Card. Numb. ★ 10·22) . ★ 11·0 . ★ 3·11 . ★ 3·12 . $\supset$. Prop].

·11 $b \supset h . \supset . \mu(b, \text{rel } P) = (2^{\mu b} - 1)^{\mu(b, \text{div } P)}$,

[★ 3·10 . ★ 10·32 . $\supset$. Prop].

·12 $\beta \,\varepsilon\, Nc \,.\supset.\, \mu\,(\beta, \mathrm{rel}\, P)^{\times} = (2^{\beta} - 1)^{\mu \smile {}'(\beta,\, \mathrm{div}\, P)}$,
[★3·11 . ★2·22 . ★11·11 . (Card. Numb. ★10·22 . ★13·1) . ⊃ . Prop].

·13 $\mu\, \mathrm{equ}\,(i, h, P) = \prod^{\beta \leqq \mu h} (2^{\beta} - 1)^{\mu \smile (\beta,\, \mathrm{div}\, P)}$,
[★11·01 . ★11·12 . ⊃ . Prop].

Note: This is the general formula for the number of relations belonging to the class equ (i, h, P), and thus also for the number of solutions of the corresponding logical equation.

·21 $\mu\{Nc \rightarrow 1 \frown \mathrm{equ}(i, h, P)\} = \mu\{\mathrm{rel}\,(Nc, Nc \rightarrow 1, P)^{\times}\} = \mu\{Nc, Nc \rightarrow 1, P\}^{\times}$,
[★4·03 . ★4·04 . ⊃ . Prop].

·22 $\mu\,\{Nc \rightarrow 1 \frown \mathrm{equ}\,(i, h, P)\} = \prod^{\beta \leqq \mu h} \mu\,(\beta, Nc \rightarrow 1, P)^{\times}$,
[(Card. Numb. ★10·22) . ⊃ . Prop].

·23 $b \supset h \,.\supset.\, \mu\,(b, Nc \rightarrow 1, P) = (\mu\, b)^{\mu\,(b,\, \mathrm{div}\, P)}$,
[★4·01 . ★3·10 . ★10·41 . ⊃ . Prop].

·24 $\beta \,\varepsilon\, Nc \,.\supset.\, \mu\,(\beta, Nc \rightarrow 1, P)^{\times} = \beta^{\smile {}'\mu\,(\beta,\, \mathrm{div}\, P)}$,
[★3·11 . ★2·22 . (Card. Numb. ★10·22 ★13·1) . ⊃ . Prop].

·25 $\mu\,\{Nc \rightarrow 1 \frown \mathrm{equ}\,(i, h, P)\} = \prod^{\beta \leqq \mu h} \beta^{\mu \smile {}'(\beta,\, \mathrm{div}\, P)}$,
[★11·22 . ★11·24 . ⊃ . Prop].

Note: This is the general formula for the number of relations belonging to the class $Nc \rightarrow 1 \frown \mathrm{equ}\,(i, h, P)$, and thus also for the number of solutions of the corresponding logical equation. M. Poretsky has given the number of solutions of a logical equation in one variable (viz., $a \frown x \smile b \frown \bar{x} = \Lambda$) in the Revue de Mathématiques, Turin, Tome VI, 1896, in his paper, "La Loi des racines en Logique." The solution given now holds for any finite or infinite number of variables. We proceed to state the propositions ★11·13 and ★11·25 in forms convenient for the case where the number of variables in the logical equations is finite; this case has already been partially considered in ★5.

★12 Hp (★5) . ⊃ ∴

·01 $\mu\, \mathrm{equ}\,(i, h, P) = \prod_{\beta > \nu}^{\beta \leqq \nu} (2^{\beta} - 1)^{\mu\,(S_{\nu-\beta} \smile \bar{S}_{\nu-\beta+1})}$,
[★5·05 . ★11·13 . ⊃ . Prop].

$\cdot 02$ $\mu \{Nc \rightarrow 1 \quad \text{equ}\,(i, h, P)\} = \prod\limits_{\beta > 1}^{\beta \leq \nu} \beta^{\mu(S_{\nu-\beta} \quad \bar{S}_{\nu-\beta+1})}$,

[$\bigstar 5{\cdot}05 . \bigstar 11{\cdot}25 . \supset .$ Prop].

Note: $\bigstar 12{\cdot}01$ and $\bigstar 12{\cdot}02$ give the number of solutions of the two types of logical equation when the number of variables is finite; $\bigstar 12{\cdot}02$ is of fundamental importance, especially in the theory of Logical Substitution Groups, developed in Section III. It can be verified (the number of variables being finite) by another method.

$\cdot 03$ $\mu \bar{S}_{\nu-1} \,\varepsilon\, Nc \text{ infin} . \supset . \mu \text{ equ}\,(i, h, P) = \mu \{Nc \rightarrow 1 \frown \text{equ}\,(i, h, P)\} = 2^{\mu \bar{S}_{\nu-1}}$

[$\alpha_2, \ldots . \alpha_\nu \,\varepsilon\, Nc . \alpha_2 + \ldots . + \alpha_\nu \,\varepsilon\, Nc \text{ infin} . \supset .$

$$\prod_{\beta > 1}^{\beta \leq \nu} (2^\beta - 1)^{\alpha_\beta} = \prod_{\beta > 1}^{\beta \leq \nu} \beta^{\alpha_\beta} = 2^{\Sigma \alpha_\beta} \qquad (1)$$

$$\begin{aligned} 1 < \beta \leq \nu . \supset . \bar{S}_{\nu-\beta} &\smile (S_{\nu-\beta} \frown \bar{S}_{\nu-\beta+1}) \\ &= \bar{S}_{\nu-\beta} \smile (\bar{S}_{\nu-\beta} \frown \bar{S}_{\nu-\beta+1}) \smile (S_{\nu-\beta} \frown \bar{S}_{\nu-\beta+1}) \\ &= \bar{S}_{\nu-\beta} \smile \bar{S}_{\nu-\beta+1} = \bar{S}_{\nu-\beta+1}. \qquad (2) \end{aligned}$$

$$\begin{aligned} (2) . S_0 = i . \supset . \mu(S_{\nu-2} \frown \bar{S}_{\nu-1}) + \mu(S_{\nu-3} \frown \bar{S}_{\nu-2}) + \ldots . \\ + \mu(S_0 \frown \bar{S}_1) = \mu \bar{S}_{\nu-1}, \qquad (4) \end{aligned}$$

Hp . (1) . (4) . $\bigstar 12{\cdot}01 . \bigstar 12{\cdot}02 . \supset .$ Prop].

Note: This proposition is a great simplification of $\bigstar 12{\cdot}01$ and of $\bigstar 12{\cdot}02$ in the most important case.

$\cdot 1$ $\mu \text{ equ}\,(i, h, P) \,\varepsilon\, Nc \text{ fin} . \mu \{Nc \rightarrow 1 \frown \text{equ}\,(i, h, P)\} \,\varepsilon\, Nc \text{ fin} : \smile :$

$\mu \bar{S}_{\nu-1} \,\varepsilon\, Nc$ infin,

[Demonst ($\bigstar 12{\cdot}03$) $. \supset .$ Prop].

Note: It follows from $\bigstar 12{\cdot}03$ and $\bigstar 12{\cdot}1$ that the number of solutions of a logical equation is either finite or is a number not less than that of the continuum.

$\cdot 2$ $2^{\mu \bar{S}_{\nu-1}} \leq \mu \{Nc \rightarrow 1 \frown \text{equ}\,(i, h, P)\} \leq \nu^{\mu \bar{S}_{\nu-1}}$,

[$\bigstar 12{\cdot}02$. (demonstration of $\bigstar 12{\cdot}03$) $. \supset .$ Prop].

Examples. (A) of $\bigstar 12{\cdot}02$,

$$(a \frown x) \smile (b \frown \bar{x}) = \Lambda.$$

Here $\nu = 2$, $S_1 = a \smile b$, $S_2 = a \frown b = \Lambda$; hence the number of solu-

tions is $2^{\mu \bar{S}_1} = 2^{\mu(\bar{a} \cap \bar{b})}$. This example is the case given by Poretsky.

(B) of ★12·02,

$$(a \cap x \cap y) \cup (b \cap x \cap \bar{y}) \cup (c \cap \bar{x} \cap y) \cup (d \cap \bar{x} \cap \bar{y}) = \Lambda.$$

Here $\nu = 4$, $S_1 = a \cup b \cup c \cup d, \ldots\ldots, \quad S_\nu = a \cap b \cap c \cap d = \Lambda$; hence the number of its solutions is

$$2^{\mu(S_2 \cap \bar{S}_3)} \times 3^{\mu(S_1 \cap \bar{S}_2)} \times 4^{\mu \bar{S}} = 2^{2 \times \mu S_1 + \mu(S_2 \cap \bar{S}_3)} \times 3^{\mu(S_1 \cap \bar{S}_2)}$$

and if $\mu \bar{S}_3$ is infinite, it follows from ★12·03 that the number of solutions can be written in the simplified form $2^{\mu \bar{S}_3}$, where

$$\bar{S}_3 = (\bar{a} \cap \bar{b}) \cup (\bar{a} \cap \bar{c}) \cup (\bar{a} \cap \bar{d}) \cup (\bar{b} \cap \bar{c}) \cup (\bar{b} \cap \bar{d}) \cup (\bar{c} \cap \bar{d}).$$

(C) of ★12·01,

$$(a \cap x_1) \cup (b \cap x_2) = \Lambda, \quad x_1 \cup x_2 = i.$$

Here $\nu = 2$, $S_1 = a \cup b$, $S_2 = a \cap b = \Lambda$; and the number of solutions is $(2^2 - 1)^{\mu \bar{S}_1} = 3^{\mu(\bar{a} \cap \bar{b})}$.

(D) of ★12·01,

$$(a \cap x_1) \cup (b \cap x_2) \cup (c \cap x_3) \cup (d \cap x_4) = \Lambda, \quad x_1 \cup x_2 \cup x_3 \cup x = \Lambda.$$[4]

Here $\nu = 4$, $S_1 = a \cup b \cup c \cup d, \ldots\ldots, S_\nu = a \cap b \cap c \cap d = \Lambda$; the number of solutions is

$$(2^2 - 1)^{\mu(S_2 \cap \bar{S}_3)} \times (2^3 - 1)^{\mu(S_1 \quad \bar{S}_2)} \times (2^4 - 1)^{\mu \bar{S}_1}.$$

Since, in ★12·02, the coefficients of the equation only enter into the answer through the invariants $S_1, \ldots\ldots S_\nu$, it follows that all equations whose left-hand sides are members of the same congruent family (cf. Symb. Log., Part II, §6), have the same number of solutions; for instance, considering an equation with two unknowns, such as that in example (B) above, for the family of secondary linear primes (cf. Symb. Log., Part I, §3), $S_1 = i$, $S_2 = i$, $S_3 = i$, $S_4 = \Lambda$; hence the number of solutions is 1, as is otherwise known (cf. Symb. Log., Part I, §3). For the family of secondary separable primes, $S_1 = i$, $S_2 = \Lambda$, $S_3 = \Lambda$, $S_4 = \Lambda$; hence the number of solutions is $3^{\mu i}$. This can be verified by considering the equation $x \cap y = \Lambda$. For the family of deficiency two and of supplemental deficiency two, $S_1 = i$, $S_2 = i$, $S_3 = \Lambda$, $S_4 = \Lambda$; and hence the number of solutions is $2^{\mu i}$. This is

immediately obvious from considering the equation (in two variables), $x \frown (y \smile \bar{y}) = \Lambda$, that is, $x = \Lambda$, and y can be any class subordinate to i.

★13 $i \,\epsilon\, \text{cls} : x \supset i \,.\, \supset_x .\, \bar{x} = i \sim x : \supset .\!\cdot\!.$

·01 $a, b \,\epsilon\, \text{cls}' i \,.\, u = z \ni [\exists\, \text{cls}' i \frown x \ni \{z = (a \frown x) \smile (b \frown \bar{x})\}] \,.\, \supset :$

$$\mu u = 2^{\mu(\bar{a} - b)},$$

$[\mu u = \mu\, \text{cls}' (\bar{a} \frown b) \,.\, (\text{Card. Numb. ★}15\cdot0) \,.\, \supset .\, \text{Prop}]$.

·11 $a \,\epsilon\, \text{cls}' i \,.\, \beta \,\epsilon\, Nc \,.\, \beta \leqq \mu a \,.\, u = \text{cls}' i \frown x \ni (x \frown a \,\epsilon\, C^i_\beta) \,.\, \supset .\, \mu u = C^{\mu a}_\beta \times 2^{\mu \bar{a}}$,

$[p = \iota C^a_\beta \smile \iota\, \text{cls}' \bar{a} \,.\, \supset .\, u = x \ni [\exists p^\times \frown m \ni (x = \smile {}^\backprime m)] \,.\, \supset .\, \text{Prop}]$.

·20 $a, b \,\epsilon\, \text{cls}' i \,.\, a \frown b = \Lambda \,.\, \alpha \,\epsilon\, Nc \,.\, \alpha \leqq \mu (a \smile b) \,.$

$u = \text{cls}' i \frown x \ni [(a \frown x) \smile (b \frown \bar{x}) \,\epsilon\, C^i_\alpha] \,.\, \supset .\, \mu u = C^{\mu(a-b)}_\alpha \times 2^{\mu p(a, \bar{b})}$.

Note: $p(a, b) = (a \frown \bar{b}) \smile (\bar{a} \smile b) \,.\, p(a, \bar{b}) = (a \frown b) \smile (\bar{a} \frown b)$ (cf. Symb. Log., Part I, §3). The proof is as follows:

$[\text{Hyp} \,.\, \supset .\, u = y \ni [\exists\, Nc \frown (\xi, \eta) \ni \{\breve{\xi} + \eta = \alpha \,.\, \exists (x_1, x_2, u) \ni (x_1 \,\epsilon\, C^a_\xi \,.$

$x_2 \,\epsilon\, C^b_\eta \,.\, u \supset i \,.\, y = x_1 \smile \bar{x}_2 \smile u \frown p(a, \bar{b}))\}] \therefore$ (1)

$(1) \,.\, (\text{Card. Numb. ★}7\cdot21) \,.\, \supset .\, \mu u = \sum_{\breve{\xi} + \eta = \alpha} C^{\mu a}_\xi \times C^{\mu b}_\eta \times 2^{\mu p(a, b)}$, (2)

$(2) \,.\, (\text{Card. Numb. ★}16\cdot1) \,.\, \supset .\, \text{Prop}]$.

·21 $a, b \,\epsilon\, \text{cls}' i \,.\, \alpha \,\epsilon\, Nc\, \text{fin} \smile N\alpha_0 \,.\, \mu(a \frown b) < \alpha \leqq \mu(a \smile b) \,.$

$u = \text{cls}' i \frown x \ni [(a \frown x) \smile (b \frown \bar{x}) \,\epsilon\, C^i_\alpha] \,.\, \supset .\, \mu u = C^{\mu p(a, b)}_{\alpha - \mu(a - b)} \times 2^{\mu p(\bar{a}, b)}$,

$[(a \frown x) \smile (b \frown \bar{x}) \,\epsilon\, C^i_\alpha \,.\, = .\, (a \frown \bar{b} \frown x) \smile (\bar{a} \frown b \frown \bar{x}) \,\epsilon\, C^i_{\alpha - \mu(a - b)}$, (1)

$(1) \,.\, ★13\cdot20 \,.\, \supset .\, \text{Prop}]$.

For the definition of $N\alpha_0$, cf. Card. Numb. ★30·0. The point of the limitation to Nc fin or to $N\alpha_0$ is that then $\alpha - \mu(a \frown b)$ is a definite number.

·22 $a, b \,\epsilon\, \text{cls}' i \,.\, \alpha \,\epsilon\, N\alpha_0 \,.\, \mu(a \frown b) < \alpha \leqq \mu(a \smile b) \,.$

$u = \text{cls}' i \frown x \ni [(a \frown x) \smile (b \frown \bar{x}) \,\epsilon\, C^i_\alpha] \,.\, \supset .$

$\mu u = C^{\mu p(a, b)}_\alpha \times 2^{\mu p(a, \bar{b})}$,

$[\alpha - \mu(a \frown b) = \alpha \,.\, ★13.21 \,.\, \supset .\, \text{Prop}]$.

·23 $a, b \,\epsilon\, \text{cls}' \iota \,.\, \alpha \,\epsilon\, Nc\, \text{fin} \,.\, \mu(a \frown b) = \alpha \,.$

$u = \text{cls}' i \frown x \ni [(a \frown x) \smile (b \frown \bar{x}) \,\epsilon\, C^i_\alpha] \,.\, \supset .\, \mu u = 2^{\mu p(a, \bar{b})}$,

$[★12\cdot02 \,.\, u = \text{cls}' i \frown x \ni [(a \frown b \frown x) \smile (\bar{a} \frown b \frown \bar{x}) = \Lambda] \,.\, \supset .\, \text{Prop}]$.

$\cdot 24\quad a, b \,\varepsilon\, \mathrm{cls}'\, i \,.\, \alpha \,\varepsilon\, N\alpha_0 \,.\, \mu(a \quad b) = \alpha \,.$
$u = \mathrm{cls}'\, i \frown x \ni [(a \frown x) \frown (b \smile \bar{x}) \,\varepsilon\, C^i_\alpha] \,.\supset .$
$$\mu u = \sum^{\nu \leqq \alpha} C_\nu^{\mu p(a, b)} \times 2^{\mu p(a, \bar{b})},$$
$[\nu \leqq \alpha \,.\supset .\, \alpha + \nu = \alpha \,.\supset .\, u = \mathrm{cls}'\, i \frown x \ni [\exists\, Nc \frown \nu \ni \{\nu \leqq \alpha \,.$
$(a \frown \bar{b} \frown x) \smile (\bar{a} \frown b \frown \bar{x}) \,\varepsilon\, C^i_\nu\}],$ (1)
$\bigstar\, 13\cdot 20 \,.\, (1) \,.\supset .\, \mathrm{Prop}] \,.$

$\cdot 30\quad a, b \,\varepsilon\, \mathrm{cls}'\, i \,.\, a \frown b = \Lambda \,.\supset .\, \sum^{x \supset i} 2^{\mu\{(a - x) \smile (b - \bar{x})\}} = 2^{\mu p(a, \bar{b})} \times 3^{\mu p(a, b)},$
$$[\bigstar\, 13\cdot 20 \,.\supset .\, \sum^{x \supset i} 2^{\mu\{(a - x) \smile (b - \bar{x})\}} = \sum^{\xi \leqq \mu(a \smile b)} C_\xi^{\mu(a \smile b)} \times 2^{\mu p(a, \bar{b})} \times 2^{\xi}$$
$$= 2^{\mu p(a, \bar{b})} \times \sum^{\xi \leqq \mu(a \smile b)} C_\xi^{\mu(a \smile b)} \times 2^{\xi}, \quad (1)$$
$(1) \,.\, (\text{Card. Numb. } \bigstar\, 17\cdot 4) \,.\, a \smile b = p(a, b) \,.\supset .\, \mathrm{Prop}] \,.$

$\cdot 31\quad a, b \,\varepsilon\, \mathrm{cls}'\, i \,.\supset .\, \sum^{x \supset i} 2^{\mu\{(a - x) \smile (b - \bar{x})\}} = 2^{\mu(a - b) + \mu p(\bar{a}, b)} \times 3^{\mu p(a, b)},$
$[2^{\mu\{(a \quad x) \smile (b - \bar{x})\}} = 2^{\mu(a - b)} \times 2^{\mu\{(a - \bar{b} - \bar{x}) \quad (a - b - \bar{x})\}},$ (1)
$p(a \frown \bar{b}, \bar{a} \frown b) = p(a, b) \,.\, p\{a \frown \bar{b}, (\bar{a} \frown b)\} = p(a, \bar{b}),$ (2)
$\bigstar\, 13\cdot 30 \,.\, (1) \,.\, (2) \,.\supset .\, \mathrm{Prop}] \,.$

SECTION III.

Orders of Various Logical Substitution Groups.

The properties of these groups have been investigated (cf. Symb. Log., Part II) for the case of functions of two variables. In the present section the orders of the various groups, discussed in the memoir referred to, will be determined. In the reference the theory of substitution groups is not investigated by symbolic methods, accordingly, these methods will be largely abandoned in the present section. A group is a class of operations and the order of the group is the cardinal number of the class. The class i will be assumed to contain the classes denoted by the two variables x and y, and also the classes denoted by the coefficients of any function of one or both of these variables. Also as before,

$$z \supset i \,.\supset_z .\, z = i \sim z.$$

The statement that $\left.\begin{matrix}\breve{\xi}_1, \breve{\xi}_2, \breve{\xi}_3, \breve{\xi}_4\\ \eta_1, \eta_2, \eta_3, \eta_4\end{matrix}\right\}$ are the coefficients of a substitution T, means that

$$Tx = (\breve{\xi}_1 \frown x \frown y) \smile (\breve{\xi}_2 \frown x \frown \bar{y}) \smile (\breve{\xi}_3 \frown \bar{x} \frown y) \smile (\breve{\xi}_4 \frown \bar{x} \frown \bar{y}),$$
$$Ty = (\eta_1 \frown x \frown y) \smile (\eta_2 \frown x \frown \bar{y}) \smile (\eta_3 \frown \bar{x} \frown y) \smile (\eta_4 \frown \bar{x} \frown \bar{y}),$$

where $\left.\begin{matrix}\breve{\xi}_1, \ldots. \breve{\xi}_4\\ \eta_1, \ldots. \eta_4\end{matrix}\right\}$ satisfy the equation (cf. Symb. Log., Part II, §2, equation (12)).

★ 20·01 $\Sigma\{(\breve{\xi}_p \frown \breve{\xi}_q) \smile (\bar{\breve{\xi}}_p \frown \bar{\breve{\xi}}_q)\} \frown \{(\eta_p \frown \eta_q) \smile (\bar{\eta}_p \frown \bar{\eta}_q)\} = \Lambda,$
$(p, q = 1, 2, 3, 4),$

which can also be written in the form

·02 $\Sigma\, \bar{p}(\breve{\xi}_p, \breve{\xi}_q) \frown \bar{p}(\eta_p, \eta_q) = \Lambda,\ (p, q = 1, 2, 3, 4),$

and also in the form

·03 $\Pi(\bar{\breve{\xi}}_r \smile \bar{\eta}_r) \smile \Pi(\bar{\breve{\xi}}_r \smile \eta_r) \smile \Pi(\breve{\xi}_r \smile \bar{\eta}_r) \smile \Pi(\breve{\xi}_r \smile \eta_r) = \Lambda,\ (r = 1, 2, 3, 4),$

and also in the form

·04 $\Pi\{(\bar{\breve{\xi}}_r \frown \eta_r) \smile (\bar{\breve{\xi}}_r \frown \bar{\eta}_r) \smile (\breve{\xi}_r \frown \bar{\eta}_r)\} \smile \Pi\{(\bar{\breve{\xi}}_r \frown \eta_r) \smile (\bar{\breve{\xi}}_r \frown \bar{\eta}_r) \smile (\breve{\xi}_r \frown \eta_r)\}$
$\smile \Pi\{(\breve{\xi}_r \frown \eta_r) \smile (\breve{\xi}_r \frown \bar{\eta}_r) \smile (\bar{\breve{\xi}}_r \frown \bar{\eta}_r)\}$
$\smile \Pi\{(\breve{\xi}_r \frown \eta_r) \smile (\breve{\xi}_r \frown \bar{\eta}_r) \smile (\bar{\breve{\xi}}_r \frown \eta_r)\} = \Lambda.$

Any one of these forms will be called the equation of condition for the coefficients of a substitution. When this equation of condition is fully developed in terms of its eight unknowns $\breve{\xi}_1 \ldots. \breve{\xi}_4, \eta_1 \ldots. \eta_4$, it has 2^8 terms; the coefficients of these various terms are either i or Λ. Considering the form ★ 20·04, it is easily seen that the first product (i. e., $\Pi\{(\bar{\breve{\xi}}_r \frown \eta_r) \smile (\bar{\breve{\xi}}_r \frown \bar{\eta}_r) \smile (\breve{\xi}_r \frown \bar{\eta}_r)\}$) gives 3^4 terms with coefficient i; the second product gives $(3^4 - 2^4)$ other terms with coefficient i, the third product gives $(3^4 - 2^4 - 2^4 + 1)$ other terms with coefficient i, the fourth product gives $(3^4 - 2^4 - 2^4 - 2^4 + 3)$ other terms with coefficient i. Hence, there are 232 terms with coefficient i and 24 with with coefficient Λ.

Thus, calculating the invariants $S_0, \ldots. S_{2^8}$ of this equation from S_0 to S_{2^8-24}, they are each equal to i, and from S_{2^8-23} to S_{2^8} they are each equal to Λ. Hence from ★ 12·02 we deduce

·1 The order of the complete logical substitution group for functions of two variables is $24^{\mu i}$. Thus, if the order is infinite, it is equal to the power of the continuum at least.

The identical group of $\phi(x, y)$ is simply isomorphic with that of the canonical function of the congruent family (cf. Symb. Log., Part II, §7) to which $\phi(x, y)$ belongs. Hence, in order to determine the order of the identical group of $\phi(x, y)$, we have only to determine it for the congruent family. Let s_1, s_2, s_3, s_4 be the invariants of this family, so that $s_4 \supset s_3 \supset s_2 \supset s_1$. Then the coefficients of a substitution of the identical group of the canonical function, in addition to satisfying the equation of condition (★20·04) must satisfy (cf. Symb. Log., Part II, §7, equ. (37)), the four equations

$$\left.\begin{array}{l}(s_1 \cap \bar{s}_2 \cap \breve{\xi}_1 \cap \bar{\eta}_1) \cup (s_1 \cap \bar{s}_3 \cap \breve{\breve{\xi}}_1 \cap \eta_1) \cup (s_1 \cap \bar{s}_4 \cap \breve{\breve{\xi}}_1 \cap \bar{\eta}_1) = \Lambda, \\ (s_1 \cap \bar{s}_2 \cap \breve{\xi}_2 \cap \eta_2) \cup (s_2 \cap \bar{s}_3 \cap \breve{\breve{\xi}}_2 \cap \eta_2) \cup (s_2 \cap \bar{s}_4 \cap \breve{\breve{\xi}}_2 \cap \bar{\eta}_2) = \Lambda, \\ (s_1 \cap \bar{s}_3 \cap \breve{\xi}_3 \cap \eta_3) \cup (s_2 \cap \bar{s}_3 \cap \breve{\xi}_3 \cap \bar{\eta}_3) \cup (s_3 \cap \bar{s}_4 \cap \breve{\breve{\xi}}_3 \cap \bar{\eta}_3) = \Lambda, \\ (s_1 \cap \bar{s}_4 \cap \breve{\xi}_4 \cap \eta_4) \cup (s_2 \cap \bar{s}_4 \cap \breve{\xi}_4 \cap \bar{\eta}_4) \cup (s_3 \cap \bar{s}_4 \cap \breve{\breve{\xi}}_4 \cap \eta_4) = \Lambda. \end{array}\right\} (1)$$

The equations can be combined with the equation of condition into one single equation of condition with 2^8 terms when fully developed. By noticing the symmetry of equations (1) and that of ★20·04, and reducing by the relations between s_1, s_2, s_3, s_4, we find that the values of the 2^8 coefficients are given by the following table, the values of the coefficients being on the upper line and the corresponding number on the lower line being the number of coefficients with that value:

i	Λ	$s_1 \cap \bar{s}_4$	$s_1 \cap \bar{s}_3$	$s_2 \cap \bar{s}_4$	$s_2 \cap \bar{s}_3$	$s_3 \cap \bar{s}_4$	$s_1 \cap \bar{s}_2$	$(s_1 \cap \bar{s}_2) \cup (s_3 \cup \bar{s}_4)$
232,	1,	13,	3,	3,	1,	1,	1,	1

Hence the values of the invariants of the equation $S_0, S_1, \ldots . S_{2^8}$ can be calculated. After some reduction we find that from $\beta = 0$ to $\beta = 2^8 - 24$ inclusive, $S_\beta = i$; and that from $\beta = 2^8 - 23$ to $\beta = 2^8 - 6$, inclusive, $S_\beta = s_1 \cap \bar{s}_4$; and that for $\beta = 2^8 - 5$ and $\beta = 2^8 - 4$, $S_\beta = s_2 \cap \bar{s}_3$; and that from $\beta = 2^8 - 3$ to $\beta = 2^8$, inclusive, $S_\beta = \Lambda$. Hence, from ★12·02, we deduce:

·2 The order of the identical group of any member of the congruent family (s_1, s_2, s_3, s_4), where $s_4 \supset s_3 \supset s_2 \supset s_1$ is

$$24^{\mu(\bar{s}_1 \frown s_4)} \times 6^{\mu\{(s_1 \frown \bar{s}_2) \smile (s_3 \frown \bar{s}_4)\}} \times 4^{\mu(s_2 \frown \bar{s}_3)},$$

which can also be written

$$12^{\mu(\bar{s}_1 \smile s_4)} \times 3^{\mu\{(s_1 \frown \bar{s}_2) \smile (s_3 \frown \bar{s}_4)\}} \times 2^{\mu(s_2 \frown \bar{s}_3)} \times 2^{\mu i}.$$

And from ★ 12·03, or from ★ 20·2, we deduce

★ 20·21 If i is an infinite class, the order of the identical group of any function is $2^{\mu i}$.

Examples. The order of the identical group of a linear secondary prime [of congruent family (i, i, i, Λ)] is $6^{\mu i}$; this is also the order for a separable secondary prime [of congruent family $(i, \Lambda, \Lambda, \Lambda)$].

The order of the identical group of a function of deficiency two and of supplemental deficiency two [of congruent family (i, i, Λ, Λ)] is $4^{\mu i}$.

If $\phi(x, y)$ and $\psi(x, y)$ be any two functions of x and y, we shall always denote by $\Theta(\phi, \psi)$ a certain important function of their coefficients, defined as follows:

Let

$$\phi(x, y) = (a_1 \frown x \frown y) \smile (a_2 \frown x \frown \bar{y}) \smile (a_3 \frown \bar{x} \frown y) \smile (a_4 \frown \bar{x} \frown \bar{y}),$$

and

$$\psi(x, y) = (b_1 \frown x \frown y) \smile (b_2 \frown x \frown \bar{y}) \smile (b_3 \frown \bar{x} \frown y) \smile (b_4 \frown \bar{x} \frown \bar{y}),$$

then $\qquad \Theta(\phi, \psi) = \Sigma\, \bar{p}(a_r, a_s;\ b_r, b_s), \qquad (r, s = 1, 2, 3, 4).$

Thus $\qquad \Theta(\phi, \psi) = \Theta(\psi, \phi).$

Also $\Theta(\phi, \psi)$ is the same as the left-hand side of ★ 20·02, after substituting a for $\breve{\xi}$ and b for η; hence, after the same substitution, $\Theta(\phi, \psi)$ can be written in the form of the left hand side of ★ 20·01, or of ★ 20·03, or of ★ 20·04. Also $\Theta(\phi, \psi) = \Lambda$ is the condition that $\phi(x, y)$ and $\psi(x, y)$ should be a pair of director-functions (cf. Symb. Log., Part II, §1) of some substitution.

We shall now prove the following theorem:

·22 If $\phi(x, y)$ and $\psi(x, y)$ are two functions of x and y, such that $\mu\Theta(\phi, \psi)$ is infinite, the order of the common subgroup of the identical groups of $\phi(x, y)$ and of $\psi(x, y)$ is $2^{\mu\Theta(\phi, \psi)}$; and otherwise the order of the subgroup is finite.

For (cf. Symb. Log., Part II, §8, equ. (37)) the coefficients $\left.\begin{matrix}\breve{\xi}_1, \breve{\xi}_2, \breve{\xi}_3, \breve{\xi}_4 \\ \eta_1, \eta_2, \eta_3, \eta_4\end{matrix}\right\}$, of any substitution of the common subgroup, satisfy in

addition to ★ 20·04, the four equations

$$\{p(a_2, a_1;\ b_2, b_1) \frown \breve{\xi}_1 \frown \bar{\eta}_1\} \smile \{p(a_3, a_1;\ b_3, b_1) \frown \breve{\bar{\breve{\xi}}}_1 \frown \eta_1\}$$
$$\smile \{p(a_4, a_1;\ b_4, b_1) \frown \breve{\bar{\breve{\xi}}}_1 \frown \bar{\eta}_1\} = \Lambda,$$
$$\{p(a_1, a_2;\ b_1, b_2) \frown \breve{\xi}_2 \frown \eta_2\} \smile \{p(a_3, a_2;\ b_3, b_2) \frown \breve{\bar{\breve{\xi}}}_2 \frown \eta_2\}$$
$$\smile \{p(a_4, a_2;\ b_4, b_2) \frown \breve{\bar{\breve{\xi}}}_2 \frown \bar{\eta}_2\} = \Lambda,$$
$$\{p(a_1, a_3;\ b_2, b_3) \frown \breve{\xi}_3 \frown \eta_3\} \smile \{p(a_2, a_3;\ b_2, b_3) \frown \breve{\xi}_3 \frown \bar{\eta}_3\}$$
$$\smile \{p(a_4, a_3;\ b_4, b_3) \frown \breve{\bar{\breve{\xi}}}_3 \frown \bar{\eta}_3\} = \Lambda,$$
$$\{p(a_1, a_4;\ b_1, b_4) \frown \breve{\xi}_4 \frown \eta_4\} \smile \{p(a_2, a_4;\ b_2, b_4) \frown \breve{\xi}_4 \frown \bar{\eta}_4\}$$
$$\smile \{p(a_3, a_4;\ b_3, b_4) \frown \breve{\bar{\breve{\xi}}}_4 \frown \eta_4\} = \Lambda.$$

Thus the complete condition, satisfied by the coefficients, is an equation in eight variables $\breve{\xi}_1 \ldots\ldots \breve{\xi}_4, \eta_1 \ldots\ldots \eta_4$ with 2^8 terms, of which $2^8 - 24$ are i, one is Λ, and the remaining 23 are equal either to single coefficients of the four equations above, or to sums of these coefficients. Now if $d_1, d_2 \ldots\ldots d_{23}$ are these remaining coefficients and $S_0, S_1, \ldots\ldots S_{2^8}$ are the invariants of the equation, it is easy to see that

$$\bar{S}_{2^8-1} = \bar{d}_1 \smile \bar{d}_2 \smile \ldots\ldots \smile \bar{d}_{23},$$

and hence

$$\bar{S}_{2^8-1} = \bar{p}(a_1, a_2;\ b_1, b_2) \smile \bar{p}(a_1, a_3;\ b_1, b_3) \smile \bar{p}(a_1, a_4;\ b_1, b_4)$$
$$\smile \bar{p}(a_2, a_3;\ b_2, b_3) \smile \bar{p}(a_2, a_4;\ b_2, b_4) \smile \bar{p}(a_3, a_4;\ b_3, b_4) = \Theta(\theta, \psi).$$

Accordingly, the required proposition follows from ★ 12·03.

Note: I have not succeeded in shortening the labor of calculating the 256 invariants $S_0, S_1, \ldots\ldots S_{2^8}$ of the complete equation satisfied by the coefficients of any substitution of the common subgroup of the identical groups of two functions. Accordingly, I have not deduced the order, when finite, of this common subgroup. But from ★ 12·2, we deduce:

★ 20·3 The order of the common subgroup of the identical groups of and two functions $\phi(x, y)$ and $\psi(x, y)$ is not less than $2^{\mu\Theta(\phi, \psi)}$ and not greater than $2^{8 \times \mu\Theta(\phi, \psi)}$.

From this we deduce, as a corollary, that if the order of this common subgroup is unity, the functions $\phi(x, y)$ and $\psi(x, y)$ are a pair of director-functions of some substitution, a proposition already known (cf. Symb. Log., Part II, §8).

★ 21·1 The cyclical group generated by any substitution T is, in general, of the 12^{th} order.

For, let

$$Tx = (a_1 \frown x \frown y) \smile (a_2 \frown x \frown \bar{y}) \smile (a_3 \frown \bar{x} \frown y) \smile (a_4 \frown \bar{x} \frown \bar{y}),$$
$$Ty = (b_1 \frown x \frown y) \smile (b_2 \frown x \frown \bar{y}) \smile (b_3 \frown \bar{x} \frown y) \smile (b_4 \frown \bar{x} \frown \bar{y}).$$

Since (cf. Symb. Log., Part II, §2),

$$a_p \frown a_q \frown a_r = \Lambda = \bar{a}_p \frown \bar{a}_q \frown \bar{a}_r, \qquad (p, q, r \text{ unequal}),$$

and $$a_p \frown a_q = a_r \frown a_s, \qquad (p, q, r, s \text{ unequal}),$$

it follows that in the complete development of i in terms of a_1, a_2, a_3, a_4 (2^4 terms), the only terms not vanishing can be written in the form $a_p \frown a_q$. Similarly for b_1, b_2, b_3, b_4.

Let $$A_{pq} = a_p \frown a_q, \quad B_{pq} = b_p \frown b_q, \qquad (p, q = 1, 2, 3, 4).$$

Then, from the condition for a substitution,

$$A_{pq} \frown B_{pq} = \Lambda, \quad A_{pq} \frown B_{rs} = \Lambda,$$

where, as in the sequel, different subscripts are unequal. Also

$$a_1 = A_{12} \smile A_{13} \smile A_{14}, \quad b_1 = B_{12} \smile B_{13} \smile B_{14},$$
$$\bar{a}_1 = A_{23} \smile A_{34} \smile A_{42}, \quad \bar{b}_1 = B_{23} \smile B_{34} \smile B_{42},$$

with similar equations for other subscripts.

Also put X_1 for $x \frown y$, X_2 for $x \frown \bar{y}$, X_3 for $\bar{x} \frown y$, X_4 for $\bar{x} \frown \bar{y}$. Then

$$X_p \frown X_q = \Lambda.$$

Hence
$$\begin{aligned}
Tx &= \Sigma A_{pq} \frown (X_p \smile X_q), && (p, q = 1, 2, 3, 4),\\
Ty &= \Sigma B_{pq} \frown (X_p \smile X_q), && \text{“} \quad \text{“}\\
T\bar{x} &= \Sigma A_{pq} \frown (X_r \smile X_s), && (p, q, r, s = 1, 2, 3, 4),\\
T\bar{y} &= \Sigma B_{pq} \frown (X_r \smile X_s), && \text{“} \quad \text{“}
\end{aligned}$$

and
$$\begin{aligned}
TX_1 &= Tx \frown Ty = \Sigma A_{pq} \frown B_{pr} \frown X_p, && \text{“} \quad \text{“}\\
TX_2 &= Tx \frown T\bar{y} = \Sigma A_{pq} \frown B_{pr} \frown X_q, && \text{“} \quad \text{“}\\
TX_3 &= T\bar{x} \frown Ty = \Sigma A_{pq} \frown B_{pr} \frown X_r, && \text{“} \quad \text{“}\\
TX_4 &= T\bar{x} \frown T\bar{y} = \Sigma A_{pq} \frown B_{pr} \frown X_s, && \text{“} \quad \text{“}
\end{aligned}$$

$$
\begin{aligned}
\text{Thus}\quad & T\{A_{pq} \frown B_{qr} \frown X_1\} = A_{pq} \frown B_{qr} \frown X_p, \quad (p, q, r, s = 1, 2, 3, 4),\\
& T\{A_{pq} \frown B_{qr} \frown X_2\} = A_{pq} \frown B_{qr} \frown X_p, \qquad \text{"} \qquad \text{"}\\
& T\{A_{pq} \frown B_{qr} \frown X_3\} = A_{pq} \frown B_{qr} \frown X_r, \qquad \text{"} \qquad \text{"}\\
& T\{A_{pq} \frown B_{qr} \frown X_4\} = A_{pq} \frown B_{qr} \frown X_s, \qquad \text{"} \qquad \text{"}
\end{aligned}
$$

Hence if P is one of the terms

$A_{pq} \frown B_{qr} \frown X_p$, or $A_{pq} \frown B_{qr} \frown X_q$, or $A_{pq} \frown B_{qr} \frown X_r$, or $A_{pq} \frown B_{qr} \frown X_s$, we can easily verify that, either $TP = P$, or $T^2P = P$, or $T^3P = P$, or $T^4P = P$; for instance,

$$
\begin{aligned}
T^3\{A_{23} \frown A_{24} \frown X_1\} &= T^2\{A_{23} \frown A_{24} \frown X_3\}\\
&= T\{A_{23} \frown A_{24} \frown X_4\} = A_{23} \frown A_{24} \frown X_1,\\
T^4\{A_{23} \frown A_{24} \frown X_3\} &= T^3\{A_{23} \frown A_{24} \frown X_4\}\\
&= T^2\{A_{23} \frown A_{24} \frown X_1\} = T\{A_{23} \frown A_{24} \frown X_2\} = A_{23} \frown A_{24} \frown X_3.
\end{aligned}
$$

Hence the smallest number n for which the equation $T^nP = P$ holds for every term P of the type defined above is 12.

But remembering the conditions satisfied by a_1, a_2, a_3, a_4 b_1, b_2, b_3, b_4, we see that we can write

$$\phi(x, y) = \Sigma g \frown A_{pq} \frown B_{qr} \frown X,$$

where $p, q, r = 1, 2, 3, 4$, g is any coefficient and X is any one of X_1, X_2, X_3, X_4. Hence the proposition follows.

SECTION IV.

The Group of Primary Prime Substitutions.

Consider a substitution T such that Tx and Ty are each functions of one variable only, not the same for both; for instance, we will suppose that Tx is a function of x only, and Ty is a function of y only. Then, if $\left.\begin{matrix}\breve{\xi}_1, \breve{\xi}_2, \breve{\xi}_3, \breve{\xi}_4\\ \eta_1, \eta_2, \eta_3, \eta_4\end{matrix}\right\}$ are the coefficients of T, we must have

$$\breve{\xi}_1 = \breve{\xi}_2,\ \breve{\xi}_3 = \breve{\xi}_4; \quad \eta_1 = \eta_3,\ \eta_2 = \eta_4,$$

and hence from ★ 20·01, $\breve{\xi}_3 = \bar{\bar{\breve{\xi}}}_1$, $\eta_2 = \bar{\eta}_1$.

Thus, $\qquad Tx = (\bar{\bar{\breve{\xi}}} \frown x) \smile (\breve{\xi} \frown \bar{x}) = p(\breve{\xi}, x),\ \ Ty = p(\eta, y),$

is the general form for such a substitution; both Tx and Ty are pri-

mary primes. Let such a substitution be called a primary prime substitution.

★ 22·0 Substitutions of the type $Tx = p(\breve{\xi}, x)$, $Ty = p(\eta, y)$ form an abelian group, in which every substitution is of the second order.

For if T' be the substitution, $T'x = p(\breve{\xi}', x)$, $T'y = p(\eta', y)$, then

$$T'Tx = \{\bar{p}(\breve{\xi}, \breve{\xi}') \frown x\} \smile \{p(\breve{\xi}, \breve{\xi}') \frown \bar{x}\},$$
$$T'Ty = \{\bar{p}(\eta, \eta') \frown y\} \smile \{p(\eta, \eta') \frown \bar{y}\}.$$

Hence $T'T$ is a substitution of the same form.

Further, these equations show that

$$TT' = T'T \text{ and } T^2 = T_0,$$

where T_0 is the identical substitution. Hence the group is abelian, and every primary prime substitution is of order two.

★ 22·1 The order of the complete group of primary prime substitutions is $4^{\mu i}$.

For whatever classes contained in i, $\breve{\xi}$ and η may be $Tx = p(\breve{\xi}, x)$ $Ty = p(\eta, y)$ belongs to the group.

·2 The class of congruent families (s_1, s_2, s_3, s_4) such that if $\phi(x, y)$ and $\psi(x, y)$ are members of the same family of the class, a primary prime substitution T can be found such that $T\phi(x, y) = \psi(x, y)$, is the class of congruent families for which $s_2 = s_3$.

For if a_1, a_2, a_3, a_4 are the coefficients of $\phi(x, y)$ and b_1, b_2, b_3, b_4 of $\psi(x, y)$, and $\breve{\xi}$, η are the parameters of the required substitution, then (cf. Symb. Log., Part II, §6, equ (31)) the condition for these two functions is

$$[\{p(a_1, b_1) \smile p(a_2, b_2) \smile p(a_3, b_3) \smile p(a_4, b_4)\} \frown \breve{\xi} \frown \eta] \smile$$
$$[\{p(a_2, b_1) \smile p(a_1, b_2) \smile p(a_4, b_3) \smile p(a_3, b_4)\} \frown \breve{\xi} \frown \bar{\eta}] \smile$$
$$[\{p(a_3, b_1) \smile p(a_4, b_2) \smile p(a_1, b_3) \smile p(a_2, b_4)\} \frown \xi \frown \eta] \smile$$
$$[\{p(a_4, b_1) \smile p(a_3, b_2) \smile p(a_2, b_3) \smile p(a_1, b_4)\} \frown \breve{\bar{\xi}} \frown \bar{\eta}] = \Lambda.$$

Now we know that the functions must be congruent, hence all we have to do is to seek the condition that any function $\phi(x, y)$ can be so transformed into the canonical function of its family; hence we may put

$$b_1 = s_1, \quad b_2 = s_2, \quad b_3 = s_3, \quad b_4 = s_4,$$

where s_1, s_2, s_3, s_4 are the invariants of the family and $s_4 \supset s_3 \supset s_2 \supset s_1$.

Thus $T\{A_{pq} \frown B_{qr} \frown X_1\} = A_{pq} \frown B_{qr} \frown X_p$, $(p, q, r, s = 1, 2, 3, 4)$,
$T\{A_{pq} \frown B_{qr} \frown X_2\} = A_{pq} \frown B_{qr} \frown X_p$, " "
$T\{A_{pq} \frown B_{qr} \frown X_3\} = A_{pq} \frown B_{qr} \frown X_r$, " "
$T\{A_{pq} \frown B_{qr} \frown X_4\} = A_{pq} \frown B_{qr} \frown X_s$, " "

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$$T^3\{A_{23} \frown A_{24} \frown X_1\} = T^2\{A_{23} \frown A_{24} \frown X_3\}$$
$$= T\{A_{23} \frown A_{24} \frown X_4\} = A_{23} \frown A_{24} \frown X_1,$$
$$T^4\{A_{23} \frown A_{24} \frown X_3\} = T^3\{A_{23} \frown A_{24} \frown X_4\}$$
$$= T^2\{A_{23} \frown A_{24} \frown X_1\} = T\{A_{23} \frown A_{24} \frown X_2\} = A_{23} \frown A_{24} \frown X_3.$$

Hence the smallest number n for which the equation $T^nP = P$ holds for every term P of the type defined above is 12.

But remembering the conditions satisfied by a_1, a_2, a_3, a_4 b_1, b_2, b_3, b_4, we see that we can write

$$\phi(x, y) = \Sigma g \frown A_{pq} \frown B_{qr} \frown X,$$

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$$\breve{\xi}_1 = \breve{\xi}_2,\ \breve{\xi}_3 = \breve{\xi}_4;\quad \eta_1 = \eta_3,\ \eta_2 = \eta_4,$$

and hence from ★20·01, $\breve{\xi}_3 = \bar{\breve{\xi}}_1$, $\eta_2 = \bar{\eta}_1$.

Thus, $Tx = (\bar{\breve{\xi}} \frown x) \smile (\breve{\xi} \frown \bar{x}) = p(\breve{\xi}, x)$, $Ty = p(\eta, y)$,

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$$T'Tx = \{\bar{p}(\breve{\xi}, \breve{\xi}') \frown x\} \smile \{p(\breve{\xi}, \breve{\xi}') \frown \bar{x}\},$$
$$T'Ty = \{\bar{p}(\eta, \eta') \frown y\} \smile \{p(\eta, \eta') \frown \bar{y}\}.$$

Hence $T'T$ is a substitution of the same form.

Further, these equations show that

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For whatever classes contained in i, $\breve{\xi}$ and η may be $Tx = p(\breve{\xi}, x)$ $Ty = p(\eta, y)$ belongs to the group.

·2 The class of congruent families (s_1, s_2, s_3, s_4) such that if $\phi(x, y)$ and $\psi(x, y)$ are members of the same family of the class, a primary prime substitution T can be found such that $T\phi(x, y) = \psi(x, y)$, is the class of congruent families for which $s_2 = s_3$.

For if a_1, a_2, a_3, a_4 are the coefficients of $\phi(x, y)$ and b_1, b_2, b_3, b_4 of $\psi(x, y)$, and $\breve{\xi}$, η are the parameters of the required substitution, then (cf. Symb. Log., Part II, §6, equ (31)) the condition for these two functions is

$$\begin{aligned}
&[\{p(a_1, b_1) \smile p(a_2, b_2) \smile p(a_3, b_3) \smile p(a_4, b_4)\} \frown \breve{\xi} \frown \eta] \smile \\
&\quad [\{p(a_2, b_1) \smile p(a_1, b_2) \smile p(a_4, b_3) \smile p(a_3, b_4)\} \frown \breve{\xi} \frown \bar{\eta}] \smile \\
&\quad [\{p(a_3, b_1) \smile p(a_4, b_2) \smile p(a_1, b_3) \smile p(a_2, b_4)\} \frown \underset{\smile}{\xi} \frown \eta] \smile \\
&\quad [\{p(a_4, b_1) \smile p(a_3, b_2) \smile p(a_2, b_3) \smile p(a_1, b_4)\} \frown \breve{\xi} \frown \bar{\eta}] = \Lambda.
\end{aligned}$$

Now we know that the functions must be congruent, hence all we have to do is to seek the condition that any function $\phi(x, y)$ can be so transformed into the canonical function of its family; hence we may put

$$b_1 = s_1, \quad b_2 = s_2, \quad b_3 = s_3, \quad b_4 = s_4,$$

where s_1, s_2, s_3, s_4 are the invariants of the family and $s_4 \supset s_3 \supset s_2 \supset s_1$.

Then the resultant of the above equation, i. e., the condition for its possibility reduces to

$$(a_1 \frown a_3 \frown \bar{a}_2 \frown \bar{a}_4) \smile (a_2 \frown a_3 \frown \bar{a}_1 \frown \bar{a}_4)$$
$$\smile (a_1 \frown a_4 \frown \bar{a}_2 \frown \bar{a}_3) \smile (a_2 \frown a_4 \frown \bar{a}_1 \frown \bar{a}_3) = \Lambda .$$

Hence, remembering that functions of the same family exist with a_1, a_2, a_3, a_4 interchanged, we find $s_2 \frown \bar{s}_3 = \Lambda$, that is, $s_2 \supset s_3$. But $s_3 \supset s_2$, hence $s_2 = s_3$.

We notice that the families (i, i, i, Λ) and $(i, \Lambda, \Lambda, \Lambda)$ both belong to this class of families.

★ 22·3 The class of congruent families (s_1, s_2, s_3, s_4), such that $s_2 = s_3$, is such that if $\phi(x, y)$ and $\psi(x, y)$ be any two members of the same family, a substitution T can be found such that

$$T\phi(x, y) = \psi(x, y) \quad \text{and} \quad T\psi(x, y) = \phi(x, y).$$

This follows from ★ 22·0 and ★ 22·2.

·4 The identical group of any function of the family (s_1, s_2, s_3, s_4) contains a primary prime subgroup of order

$$2^{\mu(s_2 \frown \bar{s}_3)} \times 4^{\mu(\bar{s}_1 - s_4)}.$$

For in the demonstration of ★ 22·2 make $\phi(x, y)$ and $\psi(x, y)$ identical by putting a_1, a_2, a_3, a_4 for b_1, b_2, b_3, b_4, then the parameters, ξ and η, of the required primary prime substitution must satisfy

$$[\{p(a_1, a_2) \smile p(a_3, a_4)\} \frown \breve{\xi} \frown \bar{\eta}] \smile [\{p(a_1, a_3) \smile p(a_2, a_4)\} \frown \bar{\bar{\xi}} \frown \eta] \smile$$
$$[\{p(a_1, a_4) \smile p(a_2, a_3)\} \frown \xi \frown \bar{\eta}] = \Lambda .$$

This equation is always possible, and if S_1, S_2, S_3, S_4 are its invariants, we find

$$S_2 \frown \breve{S}_3 = s_2 \frown \bar{s}_3, \quad S_1 \frown \breve{S}_2 = \Lambda, \quad \breve{S}_1 = \bar{s}_1 \smile s_4 .$$

Hence from ★ 12·02 the proposition follows.

July 4, 1901.

On Differential Equations Belonging to a Ternary Linearoid Group.

By F. E. Ross.

It is the object of the present paper to investigate systems of differential equations which belong to a ternary linearoid group. The investigation is confined to those cases which are essentially distinct. The last paragraph is devoted to algebraic theorems on characteristic equations. Particular cases of these theorems were first noticed in attempting to treat two-parameter groups in their unreduced form. The results thus obtained have been generalized.

Differential equations belonging to linearoid groups have been studied by E. J. Wilczynski. He has proved the existence theorem* and obtained the differential equations belonging to a binary group.†

A group of linearoid transformations is defined by the system of equations

$$\eta_i = \phi_{i1}(x;\ a_1 \dots a_r)\, y_1 + \phi_{i2}(x;\ a_1 \dots a_r)\, y_2 + \dots \\ + \phi_{in}(x;\ a_1 \dots a_r)\, y_n, \quad (i = 1, 2 \dots n), \quad (1)$$

in which $\phi_{i\kappa}$ are uniform functions of x and of $a_1 \dots a_r$, and the r parameters a_κ are essential. The corresponding differential equations are such that if $y_1 \dots y_n$ form a fundamental system of particular solutions, the general solutions are given by (1). Therefore, these solutions undergo substitutions contained in (1) when x makes circuits around the singular points of the differential equations. The study of such systems having *three* fundamental solutions is taken up in the

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* E. J. Wilczynski, "On Linearoid Differential Equations," American Journal of Mathematics, Vol. XXI, No. 4.

† E. J. Wilczynski, "On Continuous Binary Linearoid Groups and the Corresponding Differential Equations and Λ Functions," American Journal of Mathematics, Vol. XXII, No. 3.

present paper. The results obtained sufficiently indicate what is to be expected in general.

§1.—*One-Parameter Groups.*

The infinitesimal transformation of a one-parameter ternary linearoid group can be written

$$U(f) = (\psi_{11}y_1 + \psi_{12}y_2 + \psi_{13}y_3)q_1 + (\psi_{21}y_1 + \psi_{22}y_2 + \psi_{23}y_3)q_2$$
$$+ (\psi_{31}y_1 + \psi_{32}y_2 + \psi_{33}y_3)q_3, \qquad \left(q_i = \frac{\partial f}{\partial q_i}\right), \qquad (1)$$

where $\psi_{i\kappa}$ is a uniform function of x. The finite equations are obtained by integrating the linear system

$$\frac{\partial \eta_i}{\partial t} = \psi_{i1}\eta_1 + \psi_{i2}\eta_2 + \psi_{i3}\eta_3, \qquad (i = 1 \ldots . 3),$$

with the initial conditions $\eta_i = y_i$ for $t = 0$. The solutions are of the form

$$\eta_i = A_{i1}e^{\rho_1 t} + A_{i2}e^{\rho_2 t} + A_{i3}e^{\rho_3 t}, \qquad (i = 1 \ldots . 3),$$

where $\rho_1 \ldots . \rho_3$ are the roots, supposed distinct, of the cubic

$$\begin{vmatrix} \psi_{11} - \rho & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} - \rho & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} - \rho \end{vmatrix} = 0. \qquad (2)$$

This equation will be called the *characteristic equation* of the infinitesimal transformation (1).

Determining $A_{i\kappa}$ from the systems

$$\left.\begin{array}{l} (\psi_{11} - \rho_i)A_{1i} + \psi_{12}A_{2i} + \psi_{13}A_{3i} = 0, \\ \psi_{21}A_{1i} + (\psi_{22} - \rho_i)A_{2i} + \psi_{23}A_{3i} = 0, \\ \psi_{31}A_{1i} + \psi_{32}A_{2i} + (\psi_{33} - \rho_i)A_{3i} = 0, \end{array}\right\} \qquad (i = 1 \ldots . 3), \qquad (3)$$

and putting $\eta_i = y_i$ for $t = 0$, the finite equations of the group become

$$\left.\begin{aligned}
\eta_1 = \frac{1}{\Delta}\Big[&[\lambda_1(\mu_2-\mu_3)e^{\rho_1 t}+\lambda_2(\mu_3-\mu_1)e^{\rho_2 t}+\lambda_3(\mu_1-\mu_2)e^{\rho_3 t}]\,y_1\\
&+[\lambda_1(\lambda_3-\lambda_2)e^{\rho_1 t}+\lambda_2(\lambda_1-\lambda_3)e^{\rho_2 t}+\lambda_3(\lambda_2-\lambda_1)e^{\rho_3 t}]\,y_2\\
&+[\lambda_1(\lambda_2\mu_3-\lambda_3\mu_2)e^{\rho_1 t}+\lambda_2(\lambda_3\mu_1-\lambda_1\mu_3)e^{\rho_2 t}+\lambda_3(\lambda_1\mu_2-\lambda_2\mu_1)e^{\rho_3 t}]\,y_3\Big],\\
\eta_2 = \frac{1}{\Delta}\Big[&[\mu_1(\mu_2-\mu_3)e^{\rho_1 t}+\mu_2(\mu_3-\mu_1)e^{\rho_2 t}+\mu_3(\mu_1-\mu_2)e^{\rho_3 t}]\,y_1\\
&+[\mu_1(\lambda_3-\lambda_2)e^{\rho_1 t}+\mu_2(\lambda_1-\lambda_3)e^{\rho_2 t}+\mu_3(\lambda_2-\lambda_1)e^{\rho_3 t}]\,y_2\\
&+[\mu_1(\lambda_2\mu_3-\lambda_3\mu_2)e^{\rho_1 t}+\mu_2(\lambda_3\mu_1-\lambda_1\mu_3)e^{\rho_2 t}+\mu_3(\lambda_1\mu_2-\lambda_2\mu_1)e^{\rho_3 t}]\,y_3\Big],\\
\eta_3 = \frac{1}{\Delta}\Big[&[(\mu_2-\mu_3)e^{\rho_1 t}+(\mu_3-\mu_1)e^{\rho_2 t}+(\mu_1-\mu_2)e^{\rho_3 t}]\,y_1\\
&+[(\lambda_3-\lambda_2)e^{\rho_1 t}+(\lambda_1-\lambda_3)e^{\rho_2 t}+(\lambda_2-\lambda_1)e^{\rho_3 t}]\,y_2\\
&+[(\lambda_2\mu_3-\lambda_3\mu_2)e^{\rho_1 t}+(\lambda_3\mu_1-\lambda_1\mu_3)e^{\rho_2 t}+(\lambda_1\mu_2-\lambda_2\mu_1)e^{\rho_3 t}]\,y_3\Big],
\end{aligned}\right\}\quad(4)$$

where $\lambda_i = \dfrac{A_{1i}}{A_{3i}}$ and $\mu_i = \dfrac{A_{2i}}{A_{3i}}$ can be determined from (3), and where

$$\Delta = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

It is necessary and sufficient that the characteristic equation be reducible to a product of linear factors in order that (4) may generate a linearoid group, for then the coefficients of (4) are uniform functions of x.

Equations (4) no longer hold when Δ vanishes. This happens when the characteristic equation has a pair of equal roots. The question arises, can Δ vanish in any other case? This is best answered by considering (1) in its canonical form, namely:

$$U(f) = \phi_{11}y_1q_1 + (\phi_{21}y_1 + \phi_{22}y_2)q_2 + (\phi_{31}y_1 + \phi_{32}y_2 + \phi_{33}y_3)q_3,$$

a form to which it may always be reduced by a linearoid transformation. Such a transformation leaves unaltered the characteristic equation. Since it is apparent that ϕ_{11}, ϕ_{22} and ϕ_{33} are the roots of the characteristic equation corresponding to the canonical form, we have

$$\rho_i = \phi_{ii}, \qquad (i = 1, 2, 3).$$

Inserting the values of λ_i and μ_i, equations (3) become in this case

$$\left.\begin{aligned} (\phi_{11}-\rho_i)\,\lambda_i &= 0, \\ \phi_{21}\lambda_i + (\phi_{22}-\rho_i)\,\mu_i &= 0, \\ \phi_{31}\,\lambda_i + \phi_{32}\,\mu_i + \phi_{33} - \rho_i &= 0. \end{aligned}\right\} \tag{3a}$$

Supposing the roots distinct, (3a) shows that $\lambda_2 = \lambda_3 = 0$, and therefore

$$\Delta = \lambda_1\,(\mu_2 - \mu_3)\,.$$

Now λ_1 cannot vanish, for then μ_1 would vanish by (3a), which leads to the equality $\phi_{33} = \phi_{11}$, a result contrary to hypothesis. It can be shown by similar reasoning that $\mu_2 - \mu_3 \neq 0$. We therefore conclude that Δ vanishes only if the characteristic equation has at least one pair of equal roots.

The invariants of the group (4) are easily obtained. It is apparent from the form of the infinitesimal transformation that three relative linear invariants exist. Forming the expression $l\eta_1 + m\eta_2 + n\eta_3$ from (4) and applying the conditions for a relative invariant, we obtain

$$\begin{aligned} &(\mu_2-\mu_3)\,\eta_1 + (\lambda_3-\lambda_2)\,\eta_2 + (\lambda_2\mu_3-\lambda_3\mu_2)\,\eta_3 \\ &\qquad = e^{\rho_1 t}\left[(\mu_2-\mu_3)\,y_1 + (\lambda_3-\lambda_2)\,y_2 + (\lambda_2\mu_3-\lambda_3\mu_2)\,y_3\right], \\ &(\mu_3-\mu_1)\,\eta_1 + (\lambda_1-\lambda_3)\,\eta_2 + (\lambda_3\mu_1-\lambda_1\mu_3)\,\eta_3 \\ &\qquad = e^{\rho_2 t}\left[(\mu_3-\mu_1)\,y_1 + (\lambda_1-\lambda_3)\,y_2 + (\lambda_3\mu_1-\lambda_1\mu_3)\,y_3\right], \\ &(\mu_1-\mu_2)\,\eta_1 + (\lambda_2-\lambda_1)\,\eta_2 + (\lambda_1\mu_2-\lambda_2\mu_1)\,\eta_3 \\ &\qquad = e^{\rho_3 t}\left[(\mu_1-\mu_2)\,y_1 + (\lambda_2-\lambda_1)\,y_2 + (\lambda_1\mu_2-\lambda_2\mu_1)\,y_3\right], \end{aligned}$$

which may be written in the form

$$H_1 = e^{\rho_1 t}\,Y_1, \quad H_2 = e^{\rho_2 t}\,Y_2, \quad H_3 = e^{\rho_3 t}\,Y_3. \tag{5}$$

From these relative invariants can be formed the two absolute invariants

$$\vartheta_1 = Y_1^{\rho_2}\,Y_2^{-\rho_1}, \quad \vartheta_2 = Y_1^{\rho_3}\,Y_3^{-\rho_1}.$$

The three differential invariants of the first order may be obtained from (5). They are

$$\frac{d}{dx}\left(\frac{1}{\rho_i}\log Y_i\right), \qquad (i = 1,\,2,\,3).$$

We therefore have the following system of differential equations belonging to (1),

$$\left.\begin{aligned} &\frac{d}{dx}\log Y_i - \frac{1}{\rho_i}\log Y_i \frac{d\rho_i}{dx} = f_i(x), \\ &Y_1^{\rho_2} Y_2^{-\rho_1} = f_4(x), \quad Y_1^{\rho_3} Y_3^{-\rho_1} = f_5(x), \end{aligned}\right\} \tag{6}$$

where, of course, the functions $f_i(x)$ are not independent but satisfy the relations

$$\left.\begin{aligned} \frac{1}{f_4}\frac{df_4}{dx} &= \frac{1}{\rho_1\rho_2}\frac{d(\rho_1\rho_2)}{dx}\log f_4 - \rho_2 f_1 - \rho_1 f_2, \\ \frac{1}{f_5}\frac{df_5}{dx} &= \frac{1}{\rho_1\rho_3}\frac{d(\rho_1\rho_3)}{dx}\log f_5 + \rho_3 f_1 - \rho_1 f_3. \end{aligned}\right\} \tag{6a}$$

The integration of (6) will introduce one arbitrary constant. The behavior of its solutions when the independent variable makes circuits around the singular points of the system will now be found.

Assume $f_i(x)$ $(i = 1 \ldots . 5)$ to be uniform functions of x. Equations (6) give on integration

$$\log Y_i = \frac{\rho_i}{\rho_i^0}\left[c_i + \rho_i^0 \int_{x_0}^{x} \frac{f_i(x)}{\rho_i}\,dx\right], \qquad (i = 1, 2, 3), \tag{7}$$

where ρ_i^0 is the value of ρ_i for $x = x_0$. If $d_{i\kappa}$ denote the residual of $\frac{f_i(x)}{\rho_i}$ at the singular point a_κ $(\kappa = 1 \ldots . m)$, equations (7) show that $\log Y_i$ increases by $2\pi i \rho_i d_{i\kappa}$ when x makes a circuit around the singular point a_κ. Y_i therefore changes into

$$H_i = e^{2\pi i \rho_i d_{i\kappa}} Y_i. \tag{8}$$

The last two equations in (6) give, since f_4 and f_5 are uniform,

$$H_1^{\rho_2} H_2^{-\rho_1} = Y_1^{\rho_2} Y_2^{-\rho_1}, \quad \text{and} \quad H_1^{\rho_3} H_3^{-\rho_1} = Y_1^{\rho_3} Y_3^{-\rho_1},$$

which leads to the conditions

$$e^{2\pi i \rho_1\rho_2 (d_{1\kappa} - d_{2\kappa})} = 1\,; \quad e^{2\pi i \rho_1\rho_3 (d_{1\kappa} - d_{3\kappa})} = 1\,;$$

therefore, if $\rho_1\rho_2$ and $\rho_1\rho_3$ are not constants,

$$d_{1\kappa} = d_{2\kappa} = d_{3\kappa},$$

i. e., $\frac{f_1}{\rho_1}$, $\frac{f_2}{\rho_2}$, $\frac{f_3}{\rho_3}$ *must have those singular points in common at which the residuals do not vanish, and the residuals at such common points must be equal.* Suppose on the contrary $\rho_1\rho_2$ to be a constant. The first equation in (6a) becomes

$$\frac{1}{f_4}\frac{df_4}{dx} = \rho_1\rho_2\left(\frac{f_1}{\rho_1} - \frac{f_2}{\rho_2}\right).$$

Introducing $f_4(x) = e^{F(x)}$, and requiring $F(x)$ to be a uniform function, we get, on integration,

$$F(x) = \rho_1 \rho_2 \int \left(\frac{f_1}{\rho_1} - \frac{f_2}{\rho_2} \right) dx,$$

from which it follows as before that $d_{1\kappa} = d_{2\kappa}$. Other special cases can be treated similarly. If ρ_i is zero, equations (6) break down. In this case the invariant system can be formed anew from (5). In all cases (8) can be made to agree with (5), provided we put $2\pi i\, d_{i\kappa} = t$.

The cases which arise when the characteristic equation has one pair of equal roots, or all of its roots equal, require separate treatment. The investigation is carried on most easily from the canonical form of the infinitesimal transformation. The various cases have been thus treated. No results essentially different from the above were obtained. In all cases there exists a system of functions y_1, y_2, y_3 with arbitrarily assigned branch-points a_κ, undergoing an arbitrarily assigned linearoid substitution A_κ contained in the one-parameter group, when x describes a closed path around a_κ.

§ 2.—*Two-Parameter Groups.*

Two-parameter groups may be treated in a variety of ways. Direct attack leads to some interesting relations which will be noticed in another place. The only practical solution is obtained by using Lie's types of linear groups. Nineteen such types of two-parameter linear homogeneous ternary groups exist (Lie. "Continuierliche Gruppen," p. 522). If functions of x are substituted for the constants appearing in these types, all types of *linearoid* groups will be obtained, For, suppose there were a two-parameter linearoid group which could not by a linearoid transformation be reduced to one of these linearoid types. Putting $x = a$, the group becomes linear. The result is a linear group which cannot by a linear transformation be reduced to a linear type, which is impossible. Treatment of one of these types will be sufficient. The following has been selected:

$$U_1 = y_3 q_2 + \mathrm{U}; \quad U_2 = y_3 q_1 + y_1 q_2 + \phi(x) . \mathrm{U}; \quad (U_1 U_2) = 0,$$

where we have put $\mathrm{U} = y_1 q_1 + y_2 q_2 + y_3 q_3$. The general infinitesimal transformation becomes

$$[(c_1 + c_2 \phi) y_1 + c_2 y_3] q_1 + [c_2 y_1 + (c_1 + c_2 \phi) y_2 + c_1 y_3] q_2 + [(c_1 + c_2 \phi) y_3] q_3. \quad (1)$$

The finite equations are obtained by integrating the system

$$\left.\begin{aligned} \frac{d\eta_1}{dt} &= (c_1 + c_2\phi)\,\eta_1 + c_2\,\eta_3,\\ \frac{d\eta_2}{dt} &= c_2\eta_1 + (c_1 + c_2\phi)\,\eta_2 + c_1\,\eta_3,\\ \frac{d\eta_3}{dt} &= (c_1 + c_2\phi)\,\eta_3. \end{aligned}\right\} \qquad (2)$$

The roots of the characteristic equation being equal, the solution has the form

$$\eta_i = e^{\rho t}(L_i + M_i t + N_i t^2), \qquad (i = 1, 2, 3).$$

Determination of the constants in the usual way gives, after putting $t = 1$, the system of equations

$$\left.\begin{aligned} \eta_1 &= e^{c_1 + c_2\phi}\,[y_1 + c_2 y_3],\\ \eta_2 &= e^{c_1 + c_2\phi}\,[c_2 y_1 + y_2 + (c_1 + \tfrac{1}{2}c_2^2)\,y_3],\\ \eta_3 &= e^{c_1 + c_2\phi}\,y_3. \end{aligned}\right\} \qquad (3)$$

Putting in these equations $c_2 = 0$, we easily obtain the following invariants of the subgroup generated by U_1:

$$\left.\begin{aligned} &\vartheta_1 = \frac{y_1}{y_3}, \quad \vartheta_2 = y_1^{-1} e^{\frac{y_2}{y_3}},\\ &\vartheta_3 = \frac{d}{dx}\log y_1, \quad \vartheta_4 = \frac{d}{dx}\log y_3, \quad \vartheta_5 = \frac{d}{dx}\,\frac{y_2}{y_3}. \end{aligned}\right\} \qquad (4)$$

In order to obtain the invariants of group (3), it will be necessary to operate upon ϑ_i with U_2. Making use of the once extended operator

$$\begin{aligned} U_2'(f) = (\phi y_1 + y_3)\frac{\partial f}{\partial y_1} &+ (y_1 + \phi y_2)\frac{\partial f}{\partial y_2} + \phi y_3\frac{\partial f}{\partial y_3}\\ &+ (\phi y_1' + \phi' y_1 + y_3')\frac{\partial f}{\partial y_1'} + (y_1' + \phi y_2' + \phi' y_2)\frac{\partial f}{\partial y_2'} + (\phi y_3' + \phi' y_3)\frac{\partial f}{\partial y_3'}, \end{aligned}$$

we obtain after reducing

$$\left.\begin{aligned} &U_2'(\vartheta_1) = 1, \quad U_2'(\vartheta_2) = (\vartheta_1 + \vartheta_1^{-1} - \phi)\,\vartheta_2,\\ &U_2'(\vartheta_3) = \vartheta_1'\vartheta_1^{-2} + \phi', \quad U_2'(\vartheta_4) = \phi', \quad U_2'(\vartheta_5) = \vartheta_1'. \end{aligned}\right\} \qquad (5)$$

An absolute invariant must be a function of ϑ_1 and ϑ_2. Denote it by $F(\vartheta_1\,\vartheta_2)$. Applying the condition for invariance, we get

$$U_2(F) = U_2(\vartheta_1)\frac{\partial F}{\partial \vartheta_1} + U_2(\vartheta_2)\frac{\partial F}{\partial \vartheta_2} = 0.$$

Making use of (5), this becomes

$$\frac{\partial F}{\partial \vartheta_1} + (\vartheta_1 + \vartheta_1^{-1} - \phi)\,\vartheta_2\frac{\partial F}{\partial \vartheta_2} = 0,$$

the integral of which is found to be

$$F = \tfrac{1}{2}\vartheta_1 - \phi\vartheta_1 - \log(\vartheta_1\,\vartheta_2).$$

The absolute invariant of (3) becomes therefore

$$\vartheta = \tfrac{1}{2}\left(\frac{y_1}{y_3}\right)^2 - \phi\frac{y_1}{y_3} - \frac{y_2}{y_3} + \log y_3.$$

Differential invariants are found by making use of equations (5). We obtain immediately the transformation group

$$\bar{\vartheta}_1 = \vartheta_1 + t, \quad \bar{\vartheta}_4 = \vartheta_4 + \phi' t, \quad \bar{\vartheta}_5 = \vartheta_5 + \vartheta_1' t.$$

The invariants of this one-parameter group are easily found. They are ϑ_1', $\phi'\vartheta_1 - \vartheta_4$, and $\vartheta_1'\vartheta_4 - \phi'\vartheta_5$, which are the required differential invariants of (3). The differential equations sought for are therefore

$$\left.\begin{aligned} \frac{d}{dx}\left(\frac{y_1}{y_3}\right) &= f_1(x), \\ \phi'\frac{y_1}{y_3} - \frac{d}{dx}\log y_3 &= f_2(x), \\ \phi'\frac{d}{dx}\left(\frac{y_2}{y_3}\right) - \frac{d}{dx}\left(\frac{y_1}{y_3}\right)\frac{d}{dx}\log y_3 &= f_3(x), \\ \tfrac{1}{2}\left(\frac{y_1}{y_3}\right)^2 - \phi\frac{y_1}{y_3} - \frac{y_2}{y_3} + \log y_3 &= f_4(x). \end{aligned}\right\} \tag{6}$$

This system is at once seen to be of the second order. The behavior of its solutions will not be investigated in detail, but is clear in general. They will be functions uniform everywhere except in the vicinity of certain singular points, and will undergo a linearoid substitution of the two-parameter group when x describes a closed path around one of these singular points. It can be shown

that the singular points and the corresponding substitutions may be chosen arbitrarily as in the case of a one-parameter group.

§3.—*Three-Parameter Groups.*

The only groups with which we shall be concerned in this and the following paragraphs will be non-integrable groups. All those occurring so far have been integrable. The results obtained may be taken as characteristic of such groups, the differential equations arising being simple combinations of linear differential equations.

A non-integrable three-parameter group can always be supposed to have the composition

$$(U_1 U_2) = U_1, \quad (U_1 U_3) = 2U_2, \quad (U_2 U_3) = U_3.$$

A non-integrable three-parameter linearoid group reduces to a non-integrable three-parameter linear group when x is put equal to a. Let U_1, U_2, U_3 generate a non-integrable three-parameter linearoid group. The substitution $x = a$ reduces it to a non-integrable group which is the linear transform of one or the other of the groups

$$\begin{array}{llll} 1. & V_1 = y_1 q_2, & V_2 = -\tfrac{1}{2} y_1 q_1 + \tfrac{1}{2} y_2 q_2, & V_3 = -y_2 q_1, \\ 2. & W_1 = 2y_2 q_1 + y_3 q_2, & W_2 = y_1 q_1 - y_3 q_3, & W_3 = -y_1 q_2 - 2y_2 q_3, \end{array}$$

since these are the only types of non-integrable ternary linear groups. U_1, U_2 and U_3 must therefore be the linearoid transform of either 1 or 2. These groups can therefore be considered instead of the linearoid group U_1, U_2, U_3.

Type 1.—The finite equations of this group are known to be

$$\left.\begin{aligned} \eta_1 &= a y_1 + b y_2, \\ \eta_2 &= c y_1 + d y_2, \\ \eta_3 &= y_3, \end{aligned}\right\} \tag{1}$$

in which $ad - bc = 1$. There is one absolute invariant y_3. The differential invariants are

$$\vartheta_1 = y_1 y_2' - y_2 y_1', \quad \vartheta_2 = y_1 y_2'' - y_2 y_1'', \quad \vartheta_3 = y_1' y_2'' - y_2' y_1''.$$

In place of these, the system

$$\vartheta_1' = \frac{y_1 y_2'' - y_2 y_1''}{y_1 y_2' - y_2 y_1'}, \quad \vartheta_2' = \frac{y_1' y_2'' - y_2' y_1''}{y_1 y_2' - y_2 y_1'}$$

may be taken. Putting $\vartheta_1' = -f_1(x)$, $\vartheta_2' = f_2(x)$, and $y_3 = f_3(x)$, the system of invariant differential equations readily reduces to the following simple form:

$$\left.\begin{aligned} \frac{d^2 y_i}{dx^2} + f_1(x)\frac{dy_i}{dx} + f_2(x)\,y_i = 0, \\ y_3 - f_3(x) = 0, \end{aligned}\right\} \qquad (2)$$
$$(i = 1, 2).$$

In this case, therefore, the linearoid system of invariant differential equations is merely the linearoid transform of a linear system. In order that its transformation group may be the special linear group, the further condition must be imposed that ϑ_1 be invariant under the transformations of this group.

Type 2.—The finite equations are

$$\left.\begin{aligned} \eta_1 &= a^2 y_1 + 2ab y_2 + b^2 y_3, \\ \eta_2 &= ac y_1 + (ad + bc)\,y_2 + bd y_3, \\ \eta_3 &= c^2 y_1 + 2cd y_2 + d^2 y_3, \end{aligned}\right\} \qquad (3)$$

where $ad - bc = 1$. There is one absolute invariant $y_2^2 - y_1 y_3$. On account of (3), y_1, y_2 and y_3 must be solutions of a homogeneous linear differential equation of the third order. Our system becomes therefore

$$\left.\begin{aligned} \frac{d^3 y_i}{dx^3} + f_1(x)\frac{d^2 y_i}{dx^2} + f_2(x)\frac{dy_i}{dx} + f_3(x)\,y_i = 0 \quad, \\ y_2^2 - y_1 y_3 = f_4(x), \end{aligned}\right\} \qquad (4)$$
$$(i = 1, 2, 3).$$

The corresponding linearoid system is the transform of (4) under the general linearoid substitution.

§4.—*Non-Integrable r-Parameter Groups whose Simple 3-Parameter Subgroup is an Invariant Subgroup.*

By a general theorem of Engel's,* every non-integrable group of continuous transformations contains a non-integrable 3-parameter subgroup. Suppose that this is an invariant subgroup. Then the general form of all non-integrable r-parameter ternary linearoid groups containing an invariant 3-parameter simple subgroup can be obtained. As shown by Wilczynski, we can take as the non-

* Lie, "Transformationsgruppen," Vol. III, p. 757.

integrable 3-parameter subgroup any *type* of non-integrable 3-parameter linear group. There are two cases therefore to be considered, corresponding to the two types of the last paragraph.

Case 1.—The invariant simple 3-parameter subgroup is

$$U_1 = 2y_2 q_1 + y_3 q_2, \quad U_2 = y_1 q_1 - y_3 q_3, \quad U_3 = -y_1 q_2 - 2y_2 q_3. \tag{1}$$

The remaining infinitesimal transformations have the form

$$U_\kappa = \sum^{(i=1\,\ldots\,3)} (\phi^{(\kappa)}_{i1} y_1 + \phi^{(\kappa)}_{i2} y_2 + \phi^{(\kappa)}_{i3} y_3)\, q_i.$$

Since the group (1) is an invariant subgroup, we have

$$(U_i U_\kappa) = c_{i\kappa 1} U_1 + c_{i\kappa 2} U_2 + c_{i\kappa 3} U_3, \qquad (i = 1, 2, 3\,;\ \kappa = 4\ \ldots .\ r).$$

This becomes, making use of (1),

$$(U_i U_\kappa) = (c_{i\kappa 2} y_1 + 2c_{i\kappa 1} y_2)\, q_1 + (c_{i\kappa 3} y_1 - c_{i\kappa 1} y_3)\, q_2 - (2c_{i\kappa 3} y_2 + c_{i\kappa 2} y_3)\, q_3. \tag{4}$$

Equating coefficients in (4) with those obtained by direct calculation, gives the following equations of condition:

$$\left.\begin{array}{lll}
c_{12\kappa} = -2\phi^{(\kappa)}_{21} & 2c_{11\kappa} = 2(\phi^{(\kappa)}_{11} - \phi^{(\kappa)}_{22}) & 0 = \phi^{(\kappa)}_{12} - 2\phi^{(\kappa)}_{23} \\
c_{22\kappa} = 0 & 2c_{21\kappa} = -\phi^{(\kappa)}_{12} & 0 = -2\phi^{(\kappa)}_{13} \\
c_{32\kappa} = -\phi^{(\kappa)}_{12} & 2c_{31\kappa} = -\phi^{(\kappa)}_{13} & \\
c_{13\kappa} = -\phi^{(\kappa)}_{31} & 0 = 2\phi^{(\kappa)}_{21} - \phi^{(\kappa)}_{32} & c_{11\kappa} = \phi^{(\kappa)}_{22} - \phi^{(\kappa)}_{33} \\
c_{23\kappa} = \phi^{(\kappa)}_{21} & 0 = \phi^{(\kappa)}_{12} - 2\phi^{(\kappa)}_{23} & c_{21\kappa} = -\phi^{(\kappa)}_{23} \\
c_{33\kappa} = \phi^{(\kappa)}_{11} - \phi^{(\kappa)}_{55} & & c_{31\kappa} = \phi^{(\kappa)}_{13} \\
0 = \phi^{(\kappa)}_{31} & -2c_{13\kappa} = 2\phi^{(\kappa)}_{31} & -2c_{12\kappa} = \phi^{(\kappa)}_{32} \\
0 = 2\phi^{(\kappa)}_{21} - \phi^{(\kappa)}_{32} & -2c_{23\kappa} = \phi^{(\kappa)}_{32} & -2c_{22\kappa} = 0 \\
 & -2c_{33\kappa} = 2(\phi^{(\kappa)}_{22} - \phi^{(\kappa)}_{33}) & -2c_{32\kappa} = 2\phi^{(\kappa)}_{23}
\end{array}\right\} \tag{5}$$

From this system we deduce

$$\phi^{(\kappa)}_{13} = \phi^{(\kappa)}_{21} = \phi^{(\kappa)}_{31} = \phi^{(\kappa)}_{32} = 0,$$
$$\phi^{(\kappa)}_{11} - \phi^{(\kappa)}_{22} = \phi^{(\kappa)}_{22} - \phi^{(\kappa)}_{33} = c_{11\kappa}, \quad \phi^{(\kappa)}_{12} - 2\phi^{(\kappa)}_{23} = 0, \quad \phi^{(\kappa)}_{23} = -c_{21\kappa}.$$

U_κ therefore assumes the form

$$U_\kappa = \phi^{(\kappa)}_{11} y_1 q_1 + (\phi^{(\kappa)}_{11} - c_{11\kappa})\, y_2 q_2 + (\phi^{(\kappa)}_{11} - 2c_{11\kappa})\, y_3 q_3 - c_{21\kappa} (2y_2 q_1 + y_3 q_2).$$

Omitting terms of the form $c_1 U_1 + c_2 U_2 + c_3 U_3$, this becomes

$$U_\kappa = \phi_\kappa (y_1 q_1 + y_2 q_2 + y_3 q_3), \qquad (\kappa = 4\ \ldots .\ r). \tag{6}$$

The finite equations of this group are found to be

$$\left.\begin{aligned}\eta_1 &= e^{\Phi}(a^2 y_1 + 2ab y_2 + b^2 y_3),\\ \eta_2 &= e^{\Phi}(ac y_1 + (ad+bc) y_2 + bd y_3),\\ \eta_3 &= e^{\Phi}(c^2 y_1 + 2cd y_2 + d^2 y_3),\end{aligned}\right\} \tag{7}$$

where we have put $\Phi = c_4\phi_4 + \ldots + c_r\phi_r$ and $ab - bc = 1$. Group (7) possesses the relative invariant $y_2^2 - y_1 y_3$, the transformation equation being

$$\eta_2^2 - \eta_1\eta_3 = e^{2\Phi}(y_2^2 - y_1 y_3). \tag{7a}$$

Differential invariants are obtained as follows: The minors of y''', y'', y' and y in the determinant

$$\begin{vmatrix} y''' & y_1''' & y_2''' & y_3''' \\ y'' & y_1'' & y_2'' & y_3'' \\ y' & y_1' & y_2' & y_3' \\ y & y_1 & y_2 & y_3 \end{vmatrix}$$

are invariants of U_1, U_2 and U_3. Denote them by p, q, r and s respectively. An attempt will now be made to find functions of p, q, r and s which permit all the transformations of (7). Forming with this in view the three times extended operator

$$U_\kappa'''(f) = \sum_i^{1\ldots 3}\Big[\phi_\kappa y_i \frac{\partial f}{\partial y_i} + (\phi_\kappa y_i' + \phi_\kappa' y_i)\frac{\partial f}{\partial y_i'} + (\phi_\kappa y_i'' + 2\phi_\kappa' y_i' + \phi_\kappa'' y_i)\frac{\partial f}{\partial y_i''}$$
$$+ (\phi_\kappa y_i''' + 3\phi_\kappa' y_i'' + 3\phi_\kappa'' y_i' + \phi_\kappa''' y_i)\frac{\partial f}{\partial y_i'''}\Big],$$

and operating upon p, q, r and s, we finally obtain

$$\left.\begin{aligned}U_\kappa(s) &= 3\phi_\kappa s,\\ U_\kappa(p) &= 3\phi_\kappa' s + 3\phi_\kappa p,\\ U_\kappa(q) &= -3\phi_\kappa'' s + 2\phi_\kappa' p + 3\phi_\kappa q,\\ U_\kappa(r) &= \phi_\kappa''' s - \phi_\kappa'' p + \phi_\kappa' q + 3\phi_\kappa r.\end{aligned}\right\} \tag{8}$$

Integration of (8) gives the following quaternary $(r-3)$-parameter linearoid group to which p, q, r and s are subject under the transformations of the group (7):

$$\left.\begin{aligned}\bar{s} &= e^{3\Phi} s,\\ \bar{p} &= e^{3\Phi}[3\Phi' s + p],\\ \bar{q} &= e^{3\Phi}[3(\Phi'^2 - \Phi'')s + 3\Phi' p + q],\\ \bar{r} &= e^{3\Phi}[(\Phi'^3 - 3\Phi'\Phi'' + \Phi''')s + (\Phi'^2 - \Phi'')p + \Phi' q + r].\end{aligned}\right\} \tag{9}$$

From these can be obtained the relations

$$\left.\begin{aligned}\left(\frac{\bar{p}}{s}\right) &= \frac{p}{s} + 3\Phi',\\ \left(\frac{\bar{q}}{s}\right) &= \frac{q}{s} + 2\Phi'\frac{p}{s} + 3\left(\Phi'^2 - \Phi''\right),\\ \left(\frac{\bar{r}}{s}\right) &= \frac{r}{s} + \Phi'\frac{q}{s} + \left(\Phi'^2 - \Phi''\right)\frac{q}{s} + \Phi'^3 + \Phi''' - 3\Phi'\Phi'',\end{aligned}\right\} \tag{10}$$

and from (7a),

$$\log\left(\eta_2^2 - \eta_1\eta_3\right) = \log\left(y_2^2 - y_1 y_3\right) + 2\Phi. \tag{10a}$$

From (10) we form the invariants

$$\left.\begin{aligned}\vartheta_1 &= \frac{q}{s} - \tfrac{1}{3}\left(\frac{p}{s}\right)^2 + \frac{d}{dx}\left(\frac{p}{s}\right),\\ \vartheta_2 &= -\tfrac{2}{27}\left(\frac{p}{s}\right)^3 + \tfrac{1}{3}\frac{p}{s}\cdot\frac{q}{s} - \frac{r}{s} + \tfrac{1}{3}\frac{d^2}{dx^2}\left(\frac{p}{s}\right).\end{aligned}\right\} \tag{11}$$

The first equation in (10) requires that $\frac{p}{s}$ satisfy a non-homogeneous linear differential equation of order $r - 3$, the corresponding homogeneous equation having $\phi_4' \ldots . \phi_r'$ as the members of a fundamental system. The differential equations sought for become

$$\left.\begin{aligned}\frac{d^{r-3}}{dx^{r-3}}\left(\frac{p}{s}\right) + g_1(x)\frac{d^{r-4}}{dx^{r-4}}\left(\frac{p}{s}\right) + \ldots .&\\ + g_{r-4}(x)\frac{d}{dx}\left(\frac{p}{s}\right) + g_{r-3}(x)\frac{p}{s} &= f_1(x),\\ \tfrac{1}{3}\frac{d^2}{dx^2}\left(\frac{p}{s}\right) - \tfrac{2}{27}\left(\frac{p}{s}\right)^3 + \tfrac{1}{3}\frac{p}{s}\cdot\frac{q}{s} - \frac{r}{s} &= f_2(x),\\ \frac{d}{dx}\left(\frac{p}{s}\right) - \tfrac{1}{3}\left(\frac{p}{s}\right)^2 + \frac{q}{s} &= f_3(x).\end{aligned}\right\} \tag{12}$$

This system determines $\frac{p}{s}$, $\frac{q}{s}$ and $\frac{r}{s}$. We now determine y_1, y_2 and y_3 from the equation

$$\frac{d^3y}{dx^3} - \frac{p}{s}\frac{d^2y}{dx^2} + \frac{q}{s}\frac{dy}{dx} - \frac{r}{s}y = 0, \tag{13}$$

of which they are fundamental solutions. The additional equation

$$\frac{d}{dx}\log(y_2^2 - y_1 y_3) - \tfrac{2}{3}\frac{p}{s} = f_4(x), \tag{13a}$$

obtained from (10) and (10a), must also be satisfied, which condition reduces the group of equation (13) to a 3-parameter group.

It now remains to verify that the solutions of (16) belong to the r-parameter group (7), and that the constants available can be chosen so as to make these solutions undergo arbitrary substitutions of (7) when the independent variable x makes circuits around a finite number of arbitrarily assigned singular points. Let $\phi_1' \ldots\ldots \phi_{r-3}'$ be a fundamental system of solutions of the homogeneous equation corresponding to

$$\frac{d^{r-3}}{dx^{r-3}}\left(\frac{p}{s}\right) + g_1(x)\frac{d^{r-4}}{dx^{r-4}}\left(\frac{p}{s}\right) + \ldots\ldots + g_{r-3}(x)\,.\,\frac{p}{s} = f_1(x).$$

According to the theory of linear differential equations, its general solution can be written

$$\left(\frac{\bar{p}}{s}\right) = \frac{p}{s} + \sum_{\kappa}^{3\ldots\ldots r} c_\kappa \phi_\kappa' = \frac{p}{s} + 3\Phi', \tag{14}$$

where $\frac{p}{s}$ is a particular solution. If x makes circuits around arbitrarily assigned branch-points corresponding to the singular points of $f_1(x)$, supposed uniform these solutions will undergo substitutions belonging to (14). From the third equation in (12) we obtain

$$\left(\frac{q}{s}\right) = f_3(x) + \tfrac{1}{3}\left(\frac{p}{s}\right)^2 - \frac{d}{dx}\left(\frac{p}{s}\right);$$

making use of (14), the substitution for $\frac{q}{s}$ becomes, supposing $f_3(x)$ uniform

$$\left(\frac{\bar{q}}{s}\right) = f_3(x) + \tfrac{1}{3}\left(\frac{\bar{p}}{s}\right) - \frac{d}{dx}\left(\frac{\bar{p}}{s}\right) = \frac{q}{s} + 2\Phi'\,.\,\frac{p}{s} + 3\,(\Phi'^2 - \Phi''). \tag{15}$$

Substituting (14) and (15) in the second equation of (12), supposing $f_2(x)$ to be uniform, we obtain finally

$$\left(\frac{\bar{r}}{s}\right) = \left(\frac{r}{s}\right) + (\Phi'^2 - \Phi'')\,\frac{p}{s} + \Phi'\,\frac{q}{s} + \Phi'^3 + \Phi''' - 3\Phi'\Phi''. \tag{16}$$

It is clear, therefore, that the solutions of (12) undergo substitutions of the group (9) when x makes circuits around the branch-points of the system. It remains to show that the solutions of (13) in that case undergo substitutions contained in (7). Equation (13) becomes, after making the transformation $y = e^{\frac{1}{3}\int\frac{p}{s}dx}\eta$,

$$\frac{d^3\eta}{dx^3} + \vartheta_1 \frac{d\eta}{dx} + \vartheta_2 \eta = 0. \tag{17}$$

ϑ_i is given by (11). The coefficients being invariant, the solutions of (17) are subject to linear substitutions with constant coefficients,

$$\eta_i = \lambda_{i1}\eta_1 + \lambda_{i2}\eta_2 + \lambda_{i3}\eta_3, \qquad (i = 1,\ 2,\ 3), \tag{18}$$

and, therefore, after a circuit around the branch-points,

$$\bar{y}_i = e^{\frac{1}{3}\int\left(\frac{\bar{p}}{q}\right)dx}\bar{\eta}_i = e^{\frac{1}{3}\int\frac{p}{q}dx + \Phi}\left[\lambda_{i1}\eta_1 + \lambda_{i2}\eta_2 + \lambda_{i3}\eta_3\right],$$

which becomes, after change of variable,

$$\bar{y}_i = e^{\Phi}\left[\lambda_{i1}y_1 + \lambda_{i2}y_2 + \lambda_{i3}y_3\right]. \tag{19}$$

The λ's in this system are not independent. For, since y_1, y_2 and y_3 must satisfy (13a), the transformation (18) must be such as to leave $y_2^2 - y_1 y_3$ invariant. This additional condition reduces the nine-parameter group (18) to the three-parameter subgroup (7). The verification is therefore complete. For to prove that the substitutions of the subgroup (7) may be arbitrarily assigned is a problem in the theory of linear differential equations.

The second type will now be discussed. The non-integrable subgroup ${}_3G$ for this case is

$$U_1 = y_1 q_1, \quad U_2 = \tfrac{1}{2}(-y_1 q_1 + y_2 q_2), \quad U_3 = -\,y_2 q_1. \tag{20}$$

By a process similar to that used in the case of the first type, the following general form for this group was obtained:

$$U_\kappa = \phi_\kappa(x)(y_1 q_1 + y_2 q_2) + \psi_\kappa(x)\, y_3 q_3 + a_\kappa y_2 q_2, \qquad (\kappa = 4 \ldots . r). \tag{21}$$

Forming the general transformation and putting

$$\Sigma c_\kappa \phi_\kappa = \Phi; \quad \Sigma c_\kappa \psi_\kappa = \Psi; \quad \Sigma c_\kappa a_\kappa = a,$$

we obtain finally the finite equations of the group

$$\left.\begin{aligned} \eta_1 &= e^{\Phi}\left[\frac{1}{\lambda_2-\lambda_1}(\lambda_2 e^{\rho_1}-\lambda_1 e^{\rho_2})\, y_1+\frac{1}{\lambda_1-\lambda_2}(e^{\rho_1}-e^{\rho_2})\, y_2\right], \\ \eta_2 &= e^{\Phi}\left[\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}(e^{\rho_1}-e^{\rho_2})\, y_1+\frac{1}{\lambda_2-\lambda_1}(-\lambda_1 e^{\rho_1}+\lambda_2 e^{\rho_2})\, y_2\right], \\ \eta_3 &= e^{\Psi} y_3 \end{aligned}\right\} \tag{22}$$

where ρ_i are roots of the equation

$$\rho^2-a\rho-\tfrac{1}{2}ac_2-\tfrac{1}{2}c_2^2+c_1c_3=0,$$

and where λ_i is given by

$$\lambda_i^1=-\frac{c_2}{2c_3}-\frac{\rho_i}{c_3}.$$

The invariants of G_3 can be taken as

$$u=y_1y_2'-y_2y_1',\quad v=y_1y_2''-y_2y_1'',\quad w=y_1'y_2''-y_2'y_1'',\quad s=y_3. \tag{23}$$

Operating upon these with the twice-extended operator $U_\kappa''(f)$, we obtain

$$\left.\begin{aligned} &U_\kappa(u)=(2\phi_\kappa+a_\kappa)\,u, && U_\kappa(w)=-\phi''u+\phi'v+(2\phi_\kappa+a_\kappa)\,w, \\ &U_\kappa(v)=2\phi'u+(2\phi_\kappa+a_\kappa)\,v, && U_\kappa(s)=\psi_\kappa s, \end{aligned}\right\} \tag{24}$$

the integration of which gives the group induced by (22) on u, v, w and s,

$$\left.\begin{aligned} &\bar{u}=e^{2\Phi+a}.u && \bar{w}=e^{2\Phi+a}\left[(\Phi'^2-\Phi'')\,u+\Phi'v+w\right], \\ &\bar{v}=e^{2\Phi+a}\left[2\phi'u+v\right], && \bar{s}=e^{\Psi}.s, \end{aligned}\right\} \tag{25}$$

and, therefore,

$$\left(\frac{\bar{v}}{u}\right)=\frac{v}{u}+2\Phi',\quad \log\bar{s}=\log s+\Psi, \tag{26}$$

leading to the absolute invariant

$$\vartheta_1=\tfrac{1}{2}\frac{d}{dx}\left(\frac{v}{u}\right)-\tfrac{1}{4}\left(\frac{v}{u}\right)^2+\frac{w}{u}. \tag{27}$$

According to (26), $\frac{v}{u}$ satisfies a non-homogeneous linear differential equation of order $r-3$. The system of differential equations belonging to (22) becomes therefore

$$\left.\begin{aligned} \frac{d^{r-3}}{dx^{r-3}}\left(\frac{v}{u}\right)+g_1(x)\frac{d^{r-4}}{dx^{r-4}}\left(\frac{v}{u}\right)+\dots+g_{r-3}(x)\left(\frac{v}{u}\right)&=f_1(x), \\ \tfrac{1}{2}\frac{d}{dx}\left(\frac{v}{u}\right)-\tfrac{1}{4}\left(\frac{v}{u}\right)^2+\frac{w}{u}&=f_2(x). \end{aligned}\right\} \tag{28}$$

Since $\log y_3$ undergoes the same substitutions as $\frac{v}{u}$, it is determined by an equation of the form

$$\log y_3 = h_0(x) + h_1(x)\frac{v}{u} + h_2(x)\frac{d}{dx}\frac{v}{u} + \ldots + h_{r-3}(x)\frac{d^{r-4}}{dx^{r-4}}\left(\frac{v}{u}\right), \quad (29)$$

in which $h_1 \ldots h_{r-3}$ are uniform functions, determined by the system of equations

$$h_1\phi_i^{(1)} + h_2\phi_i^{(2)} + \ldots + h_{r-3}\phi_i^{r-3} = \psi_i, \qquad (i = 4 \ldots r), \quad (29a)$$

and where h_0 is an arbitrary uniform function. $\frac{v}{u}$ and $\frac{w}{u}$ having been found from (28), y_1 and y_2 are determined from

$$\frac{d^2y_i}{dx^2} - \frac{v}{u}\frac{dy_i}{dx} + \frac{w}{u}y_i = 0, \qquad (i = 1, 2) \quad (30)$$

The behavior of the functions y_1, y_2 and y_3, as defined by (28), (29) and (30), is obtained as in the first case. The results are essentially the same, and will therefore not be given.

§5.—*Non-Integrable r-Parameter Groups whose Simple Three-Parameter Subgroup is not an Invariant Subgroup.*

4-Parameter Groups.—There are no 4-parameter groups of this class. For we have (Lie, Tr. Gr., III, p. 723) for all non-integrable 4-parameter groups the one typical composition $(U_i U_4) = 0$, where $U_i\,(i = 1, 2, 3)$ are the infinitesimal transformations of the simple 3-parameter subgroup.

5-Parameter Groups.—All groups of this kind have the same composition as (Tr. Gr., III, p. 736)

$$U_1 = xq, \quad U_2 = xp - yq, \quad U_3 = yp, \quad U_4 = p, \quad U_5 = q. \quad (1)$$

Assuming G_3 to be of the first form or

$$U_1 = y_1q_2, \quad U_2 = y_1q_1 - y_2q_2, \quad U_3 = y_2q_1,$$

we obtain the following linearoid types, having the same composition as (1):

$$\left.\begin{array}{lll} \text{A.} & G_3, & U_4 = \phi(x)\,y_3q_1, \quad U_5 = \phi(x)\,y_3q_2. \\ \text{B.} & G_3, & U_4 = \phi(x)\,y_2q_3, \quad U_5 = \phi(x)\,y_1q_3. \end{array}\right\} \quad (2)$$

The finite equations of group A are, as may be easily verified,

$$\left.\begin{aligned}\eta_1 &= ay_1 + by_2 + e\phi y_3,\\ \eta_2 &= cy_1 + dy_2 + f\phi y_3,\\ \eta_3 &= y_3,\end{aligned}\right\} \tag{3}$$

where a, b, c, d, e, f are constants subject to the condition

$$ad - bc = 1,$$

which, moreover, we may suppress, thus obtaining a 6-parameter group. Put

$$\frac{1}{\phi}\,\frac{y_1}{y_3} = Y_1, \quad \frac{1}{\phi}\,\frac{y_2}{y_3} = Y_2. \tag{4}$$

Then Y_1 and Y_2 are transformed by the general linear group

$$\left.\begin{aligned}H_1 &= aY_1 + bY_2 + e,\\ H_2 &= cY_1 + dY_2 + f,\end{aligned}\right\} \tag{5}$$

so that Y_1', Y_2' are transformed by the general linear homogeneous group. Y_1', Y_2' therefore constitute a fundamental system of solutions of a linear homogeneous differential equation of the second order, and Y_1, Y_2 themselves are integrals of such functions, while y_3 is itself a uniform function of x.

The finite equation of group B may be written:

$$\left.\begin{aligned}\eta_1 &= ay_1 + by_2,\\ \eta_2 &= cy_1 + dy_2,\\ \eta_3 &= e\phi y_1 + f\phi y_2 + y_3.\end{aligned}\right\} \tag{6}$$

Therefore, y_1, y_2 form a fundamental system of a linear differential equation of the second order, say

$$y'' + py' + qy = 0, \tag{7}$$

while $\frac{1}{\phi}\,y_3$ is a solution of the non-homogeneous linear differential equation

$$y'' + py' + qy = r, \tag{8}$$

whose left member is identical with the left member of (7). For if Y denotes any solution of (8), its general solution is

$$H = Y + ey_1 + fy_2,$$

where e and f are two arbitrary constants.

If G_3 is assumed to be of the second form

$$U_1 = 2y_2 q_1 + y_3 q_2, \quad U_2 = -2y_1 q_1 + 2y_3 q_3, \quad U_3 = y_1 q_2 + 2y_2 q_3,$$

it will be found that there are no 5-parameter groups of this class. By making use of Lie's types of 6-parameter non-integrable groups, the same was found to be true of 6-parameter groups also.

The results obtained for 5-parameter groups of this class sufficiently indicate what is to be expected in general. The study of r-parameter groups belonging to this class will therefore not be followed out any further.

From the preceding investigation, it appears that finite linearoid groups do not succeed in defining any essentially new functions. It has been shown by Wilczynski* that higher transcendental functions exist, whose multiformity is qualitatively of the same kind as that of the functions occurring in this paper Their group, however, is not contained in any finite continuous linearoid group. It is very doubtful if they satisfy any algebraic differential equations.

§6.—*On the Characteristic Equation belonging to Certain Linear and Linearoid Groups.*

The following theorems about characteristic equations, although most easily stated in terms of group theory, are of a purely algebraic nature. Consider a two-parameter group with the composition

$$(U_1 U_2) = aU_1, \tag{1}$$

where $a \neq 0$, and may, moreover, without loss of generality, be taken equal to unity. Writing

$$\left.\begin{aligned} U_1 &= (\phi_{11} y_1 + \phi_{12} y_2) q_1 + (\phi_{21} y_1 + \phi_{22} y_2) q_2, \\ U_2 &= (\psi_{11} y_1 + \psi_{12} y_2) q_1 + (\psi_{21} y_1 + \psi_{22} y_2) q_2, \end{aligned}\right\} \tag{2}$$

we derive from (1)

$$\left.\begin{aligned} A_1 &\equiv \phi_{21}\psi_{12} - \phi_{12}\psi_{21} &&= \phi_{11}, \\ A_2 &\equiv \phi_{12}\psi_{11} - \phi_{11}\psi_{12} + \phi_{22}\psi_{12} - \phi_{12}\psi_{22} &&= \phi_{12}, \\ A_3 &\equiv \phi_{11}\psi_{21} - \phi_{21}\psi_{11} + \phi_{21}\psi_{22} - \phi_{22}\psi_{21} &&= \phi_{21}, \\ A_4 &\equiv \phi_{12}\psi_{21} - \phi_{21}\psi_{12} &&= \phi_{22}, \end{aligned}\right\} \tag{3}$$

* American Journal of Mathematics, Vol. XXI, No. 3.

where A_i stands for the left members of these equations. The following identities are easily verified:

$$\left.\begin{aligned} A_1 + A_4 &= 0, \\ \phi_{21} A_2 + \phi_{12} A_3 + (\phi_{11} - \phi_{22}) A_1 &= 0, \\ \psi_{21} A_2 + \psi_{12} A_3 + (\psi_{11} - \psi_{22}) A_1 &= 0, \end{aligned}\right\} \tag{4}$$

from which we derive

$$\phi_{11} + \phi_{22} = 0, \quad (\phi_{11} - \phi_{22})\,\phi_{11} + 2\phi_{12}\,\phi_{21} = 0.$$

The coefficients of the characteristic equation belonging to U_1,

$$\rho^2 - (\phi_{11} + \phi_{22})\,\rho + \phi_{11}\,\phi_{22} - \phi_{12}\,\phi_{21} = 0$$

are therefore zero. Both of its roots are therefore zero.*

In the ternary case the system corresponding to (3) becomes

$$\left.\begin{aligned}
A_1 &\equiv \phi_{21}\psi_{12} - \phi_{12}\psi_{21} + \phi_{31}\psi_{13} - \phi_{13}\psi_{31} &&= \phi_{11}, \\
A_2 &\equiv \phi_{12}\psi_{11} - \phi_{11}\psi_{12} + \phi_{22}\psi_{12} - \phi_{12}\psi_{22} + \phi_{32}\psi_{13} - \phi_{13}\psi_{32} &&= \phi_{12}, \\
A_3 &\equiv \phi_{13}\psi_{11} - \phi_{11}\psi_{13} + \phi_{23}\psi_{12} - \phi_{12}\psi_{23} + \phi_{33}\psi_{13} - \phi_{13}\psi_{33} &&= \phi_{13}, \\
A_4 &\equiv \phi_{11}\psi_{21} - \phi_{21}\psi_{11} + \phi_{21}\psi_{22} - \phi_{22}\psi_{21} + \phi_{31}\psi_{23} - \phi_{23}\psi_{31} &&= \phi_{21}, \\
A_5 &\equiv \phi_{12}\psi_{21} - \phi_{21}\psi_{12} + \phi_{32}\psi_{23} - \phi_{23}\psi_{32} &&= \phi_{22}, \\
A_6 &\equiv \phi_{13}\psi_{21} - \phi_{21}\psi_{13} + \phi_{23}\psi_{22} - \phi_{22}\psi_{23} + \phi_{33}\psi_{23} - \phi_{23}\psi_{33} &&= \phi_{23}, \\
A_7 &\equiv \phi_{11}\psi_{31} - \phi_{31}\psi_{11} + \phi_{21}\psi_{32} - \phi_{32}\psi_{21} + \phi_{31}\psi_{33} - \phi_{33}\psi_{31} &&= \phi_{31}, \\
A_8 &\equiv \phi_{12}\psi_{31} - \phi_{31}\psi_{12} + \phi_{22}\psi_{32} - \phi_{32}\psi_{22} + \phi_{32}\psi_{33} - \phi_{33}\psi_{32} &&= \phi_{32}, \\
A_9 &\equiv \phi_{13}\psi_{31} - \phi_{31}\psi_{13} + \phi_{22}\psi_{32} - \phi_{32}\psi_{22} &&= \phi_{33},
\end{aligned}\right\} \tag{5}$$

from which we obtain the identities

$$\left.\begin{aligned}
& A_1 + A_5 + A_9 \equiv 0, \\
& \phi_{21} A_2 + \phi_{12} A_4 + \phi_{31} A_3 + \phi_{13} A_7 + \phi_{32} A_6 + \phi_{23} A_8 \\
& \qquad + (\phi_{11} - \phi_{33}) A_1 + (\phi_{22} - \phi_{33}) A_5 \equiv 0, \\
& (\phi_{22}\phi_{33} - \phi_{23}\phi_{32}) A_1 + (\phi_{23}\phi_{31} - \phi_{21}\phi_{33}) A_2 + (\phi_{21}\phi_{32} - \phi_{22}\phi_{31}) A_3 \\
& \quad + (\phi_{13}\phi_{32} - \phi_{12}\phi_{33}) A_4 + (\phi_{11}\phi_{33} - \phi_{13}\phi_{31}) A_5 + (\phi_{12}\phi_{31} - \phi_{11}\phi_{32}) A_6 \\
& \quad + (\phi_{12}\phi_{23} - \phi_{22}\phi_{13}) A_7 + (\phi_{13}\phi_{21} - \phi_{11}\phi_{23}) A_8 + (\phi_{11}\phi_{22} - \phi_{12}\phi_{21}) A_9 \equiv 0,
\end{aligned}\right\} \tag{6}$$

and two others, differing from these only in having ψ in place of ϕ. Making use of (5), we find

* This result was derived by Wilczynski (American Journal, Vol. XXII, No. 3, p. 208), who also noticed there that the roots of the characteristic equation of $U_2(f)$ differ by unity.

$$\left.\begin{aligned} \phi_{11} + \phi_{22} + \phi_{33} = 0, \\ (\phi_{11} - \phi_{33})\phi_{11} + (\phi_{22} - \phi_{33})\phi_{22} + 2(\phi_{12}\phi_{21} + \phi_{13}\phi_{31} + \phi_{23}\phi_{32}) = 0, \\ \phi_{11}\phi_{22}\phi_{33} + \phi_{21}\phi_{32}\phi_{13} + \phi_{31}\phi_{23}\phi_{12} - \phi_{13}\phi_{22}\phi_{31} - \phi_{23}\phi_{32}\phi_{11} - \phi_{12}\phi_{21}\phi_{33} = 0. \end{aligned}\right\} \quad (7)$$

Combining the first equation with the second, the left members, it will be noticed, agree with the coefficients of the characteristic equation belonging to U_1, namely,

$$\rho^3 - (\phi_{11} + \phi_{22} + \phi_{33})\rho^2 + (\phi_{11}\phi_{33} + \phi_{22}\phi_{33} + \phi_{11}\phi_{22} - \phi_{13}\phi_{31} - \phi_{32}\phi_{23} - \phi_{12}\phi_{21})\rho$$
$$- \phi_{11}\phi_{22}\phi_{33} + \phi_{21}\phi_{32}\phi_{13} + \phi_{31}\phi_{23}\phi_{12} - \phi_{13}\phi_{22}\phi_{31} - \phi_{23}\phi_{32}\phi_{11} - \phi_{12}\phi_{21}\phi_{33} = 0,$$

from which the theorem follows for this case also. The general case will now be considered.

A group of composition (1) is integrable. U_1 and U_2 can therefore be transformed simultaneously by a linearoid transformation into the canonical form

$$\left.\begin{aligned} U_1 = \phi_{11} y_1 q_1 + (\phi_{21} y_1 + \phi_{22} y_2) q_2 + (\phi_{31} y_1 + \phi_{32} y_2 + \phi_{33} y_3) q_3 + \cdots \\ U_2 = \psi_{11} y_1 q_1 + (\psi_{21} y_1 + \psi_{22} y_2) q_2 + (\psi_{31} y_1 + \psi_{32} y_2 + \psi_{33} y_3) q_3 + \cdots \end{aligned}\right\} \quad (8)$$

Such a transformation does not alter the characteristic equation or the composition of the group.

Making use of equation (1), we obtain

$$\begin{aligned} & (\psi_{11} q_1 + \psi_{21} q_2 + \cdots + \psi_{n1} q_n)\phi_{11} y_1 \\ & + (\psi_{22} q_2 + \psi_{32} q_3 + \cdots + \psi_{n2} q_n)(\phi_{21} y_1 + \phi_{22} y_2) \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + (\psi_{n-1,n-1} q_{n-1} + \psi_{n,n-1} q_n)(\phi_{n-1,1} y_1 + \phi_{n-1,2} y_2 + \cdots + \phi_{n-1,n-1} y_{n-1}) \\ & + \psi_{nn} q_n (\phi_{n1} y_1 + \cdots + \phi_{nn} y_n) \\ & - [\text{same expression with } \phi \text{ and } \psi \text{ interchanged}] \\ & = \phi_{11} y_1 q_1 + (\phi_{21} y_1 + \phi_{22} y_2) + \cdots + (\phi_{n1} y_1 + \cdots + \phi_{nn} y_n) q_n. \end{aligned} \quad (9)$$

Equating coefficients of $y_\kappa q_\kappa$ gives

$$\phi_{\kappa\kappa} = 0, \qquad (\kappa = 1 \ldots n), \quad (10)$$

and equating coefficients of $y_{\kappa-1} q_\kappa$,

$$\phi_{\kappa-1,\kappa-1}\psi_{\kappa,\kappa-1} + \phi_{\kappa,\kappa-1}\psi_{\kappa\kappa} - \phi_{\kappa,\kappa-1}\psi_{\kappa-1,\kappa-1} - \phi_{\kappa\kappa}\psi_{\kappa,\kappa-1} = \phi_{\kappa,\kappa-1},$$

which becomes, making use of (10),

$$\phi_{\kappa,\kappa-1}(\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1}) = \phi_{\kappa,\kappa-1},$$

and therefore

$$\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1} = 1 \text{ if } \phi_{\kappa,\kappa-1} \neq 0. \quad (11)$$

Noticing that $\phi_{\kappa\kappa}$ and $\psi_{\kappa\kappa}$ $(\kappa = 1 \ldots . n)$ are respectively the roots of the characteristic equations belonging to U_1 and U_2, (10) and (11) give the following theorem:

If U_1 and U_2 generate a two-parameter linear or linearoid group having the composition $(U_1 U_2) = U_1$, the characteristic equation belonging to U_1 will have all of its roots zero, while the roots of the characteristic equation belonging to U_2 will form an arithmetical progression with the common difference unity, provided that none of the coefficients of form $\phi_{\kappa,\kappa-1}$ are absent.

Unless all of the quantities $\phi_{\kappa,\kappa-1}$ are zero, there is at least one pair of roots differing by unity. If, on the contrary, these quantities are all zero, we obtain by equating coefficients of $y_{\kappa-2,\kappa}$,

$$\phi_{\kappa-2,\kappa-2}\psi_{\kappa,\kappa-2} + \phi_{\kappa-1,\kappa-2}\psi_{\kappa,\kappa-1} + \phi_{\kappa,\kappa-2}\psi_{\kappa\kappa} - \phi_{\kappa,\kappa-2}\psi_{\kappa-2,\kappa-2} - \phi_{\kappa,\kappa-1}\psi_{\kappa-1,\kappa-2} - \phi_{\kappa\kappa}\psi_{\kappa,\kappa-2} = \phi_{\kappa,\kappa-2}.$$

According to the hypothesis, all the terms in the left member vanish except two. Therefore,

$$\phi_{\kappa,\kappa-2}(\psi_{\kappa\kappa} - \psi_{\kappa-2,\kappa-2}) = \phi_{\kappa,\kappa-2},$$

which proves, as before, that there is at least one pair of roots differing by unity unless all of the coefficients $\phi_{\kappa,\kappa-2}$ also vanish. Suppose more generally that all of the quantities $\phi_{\kappa,\kappa}$, $\phi_{\kappa,\kappa-1}$, $\phi_{\kappa,\kappa-2}$, ... $\phi_{\kappa,\kappa-s}$ vanish. Equating coefficients of $y_{\kappa-s-1}\, q_\kappa$ in (9), we obtain

$$\begin{aligned}\phi_{\kappa-s-1,\kappa-s-1}\psi_{\kappa,\kappa-s-1} &+ \phi_{\kappa-s,\kappa-s-1}\psi_{\kappa,\kappa-s} + \phi_{\kappa-s+1,\kappa-s-1}\psi_{\kappa,\kappa-s+1} + \cdots \\ &+ \phi_{\kappa-1,\kappa-s-1}\psi_{\kappa,\kappa-1} + \phi_{\kappa,\kappa-s-1}\psi_{\kappa\kappa} \\ &- \phi_{\kappa,\kappa-s-1}\psi_{\kappa-s-1,\kappa-s-1} - \phi_{\kappa,\kappa-s}\psi_{\kappa-s,\kappa-s-1} - \phi_{\kappa,\kappa-s+1}\psi_{\kappa-s+1,\kappa-s-1} - \cdots \\ &- \phi_{\kappa,\kappa-1}\psi_{\kappa-1,\kappa-s-1} - \phi_{\kappa\kappa}\psi_{\kappa,\kappa-s-1} = \phi_{\kappa,\kappa-s-1}.\end{aligned} \tag{12}$$

Taking into account the assumption made, this reduces to

$$\phi_{\kappa,\kappa-s-1}(\psi_{\kappa\kappa} - \psi_{\kappa-s-1,\kappa-s-1}) = \phi_{\kappa,\kappa-s-1}. \tag{13}$$

Unless, therefore, every $\phi_{\kappa,\kappa-s}$ $(\kappa = 1 \ldots . n;\ s = 0 \ldots . \kappa - 1)$ vanishes, there will be at least one pair of roots differing by unity. All of the $\phi_{\kappa,\kappa-s}$'s cannot vanish, for this means that the infinitesimal transformation U_1 vanishes. Hence we conclude

The characteristic equation belonging to U_2 always has at least one pair of roots differing by unity.

The theorem that the characteristic equation belonging to $U_1(f)$ has only vanishing roots, is not to be confounded with a theorem proved by Killing which, on the surface, appears to be the same.* The characteristic equation employed by Killing is quite different from the one considered here.

The second part of the theorem can be read as a property of the discriminant of the characteristic equation belonging to U_2, namely: If U_1 and U_2 generate a two-parameter group of composition $(U_1 U_2) = a U_1$, the discriminant of the characteristic equation belonging to U_2 has in the general case the numerical value $\frac{[2.3.4 \ldots . (n-1)]^{2n}}{[2^2 3^3 4^4 \ldots . (n-1)^{n-1}]^2} a^{n(n-1)}$. More generally, all of the invariants of this equation are numerical, since they are functions of the differences of the roots.

§7.—*General Theorem.*

The theorem, proved in the last paragraph, can be immediately extended to r-parameter integrable groups. The infinitesimal transformations of an integrable linear or linearoid group can be chosen to satisfy the equation (Lie, "Continuierliche Gruppen," p. 537),

$$(U_i U_{i+\varkappa}) = \sum_{1}^{i+\varkappa-1} c_{i,\, i+\varkappa,\, s} U_s, \qquad \begin{pmatrix} i = 1, 2 \ldots . r \\ \varkappa = 1, 2 \ldots . r - i \end{pmatrix}. \tag{1}$$

The infinitesimal transformations can be put into the form

$$U_\varkappa = \phi_{11}^{(\varkappa)} y_1 q_1 + (\phi_{21}^{(\varkappa)} y_1 + \phi_{22}^{(\varkappa)} y_2) q_2 + \ldots . + (\phi_{n1}^{(\varkappa)} y_1 + \ldots . + \phi_{nn}^{(\varkappa)} y_n) q_n. \tag{2}$$

If we assume

$$(U_i U_\varkappa) = c_{i\varkappa 1} U_1 + c_{i\varkappa 2} U_2 + \ldots . + c_{i\varkappa r} U_r, \tag{3}$$

we obtain, by equating coefficients of $y_\lambda q_\lambda$, the system of equations

$$c_{i\varkappa 1} \phi_{\lambda\lambda}^{(1)} + c_{i\varkappa 2} \phi_{\lambda\lambda}^{(2)} + \ldots . + c_{i\varkappa r} \phi_{\lambda\lambda}^{(r)} = 0, \qquad (i, \varkappa, \lambda = 1 \ldots . r). \qquad 4)$$

Putting $i = \varkappa = 1$, equations (1) become

$$(U_1 U_2) = c_{121} U_1.$$

For this case equations (4) become

$$c_{121} \phi_{\lambda\lambda}^{(1)} = 0, \qquad (\lambda = 1 \ldots . r), \tag{5}$$

* Lie, "Transformationsgruppen," Vol. III, p. 772.

hence, if $c_{121} \neq 0$, all of the roots of the characteristic equation belonging to U_1 are zero. We also have from (1)

$$\left.\begin{aligned}(U_1 U_3) &= c_{131} U_1 + c_{132} U_2, \\ (U_2 U_3) &= c_{231} U_1 + c_{232} U_2.\end{aligned}\right\}$$

Equations (4) become

$$\left.\begin{aligned}c_{131} \phi^{(1)}_{\lambda\lambda} + c_{132} \phi^{(2)}_{\lambda\lambda} &= 0, \\ c_{231} \phi^{(1)}_{\lambda\lambda} + c_{232} \phi^{(2)}_{\lambda\lambda} &= 0,\end{aligned}\right\}$$

from which it follows that $\phi^{(2)}_{\lambda\lambda} = 0$, if we assume the roots of U_1 all zero, and c_{132} and c_{232} not both zero. The general theorem is clear, and may be stated as follows:

If all of the roots of the characteristic equations belonging to the infinitesimal transformations $U_1 \ldots U_\kappa$ vanish, where these are the κ infinitesimal transformations of a κ-parameter invariant subgroup of an integrable group having the composition (1), *the same is true of the characteristic equation belonging to $U_{\kappa+1}$, provided that not all of the composition constants*

$$c_{1,\kappa+2,\kappa+1}, \quad c_{2,\kappa+2,\kappa+1}, \ \ldots\ldots \ c_{\kappa+1,\kappa+2,\kappa+1}$$

are equal to zero.

If these conditions are not fulfilled, we proceed as follows: Suppose the roots of the characteristic equation belonging to U_μ have been found by the method outlined above to be zero. Strike out the μ^{th} term in system (4) for each value of μ. If there is left any equation containing only a single term $c_{i\kappa\tau} \phi^{(\tau)}_{\lambda\lambda}$, we conclude that the roots of the characteristic equation belonging to U_τ are zero. We would then drop the τ^{th} term from the equations, and repeat the process until we obtained a system of equations each one of which consists of two or more terms. In this way we find all of the infinitesimal transformations whose characteristic equations have zero roots.

The second part of the theorem contained in the last paragraph can also be extended. It can be stated as follows:

If the characteristic equations belonging to the infinitesimal transformations of a p-parameter invariant subgroup of an r-parameter integrable linear or linearoid group have all their roots zero, and the roots of the characteristic equation belonging to U_{p+1} are not zero, then the differences between its roots are constants depending only upon the composition of the group, and no more than p of these differences are distinct, provided that not all the terms in $U_1 \ldots\ldots U_p$ of the form $\phi^{(\kappa)}_{\lambda,\lambda-1} y_{\lambda-1} q_\lambda$ ($\lambda = 1 \ldots\ldots n;\ \kappa = 1 \ldots\ldots p$) *vanish. If all of these terms for μ distinct values*

of λ vanish, the theorem is still true for those differences which correspond to values of λ different from these. And in any case there will be at least one pair of roots differing by a constant.

The proof of this theorem is similar to that employed in the last paragraph for the special case there considered. Writing

$$\left.\begin{aligned} U_i &= \phi_{11}^{(i)} y_1 q_1 + (\phi_{21}^{(i)} y_1 + \phi_{22}^{(i)} y_2) q_2 + \dots + (\phi_{n1}^{(i)} y_1 + \dots + \phi_{nn}^{(i)} y_n) q_n, \\ U_{p+1} &= \psi_{11} y_1 q_1 + (\psi_{21} y_1 + \psi_{22} y_2) q_2 + \dots + (\psi_{n1} y_1 + \dots + \psi_{nn} y_n) q_n, \end{aligned}\right\} \quad (6)$$
$$(i = 1 \dots p),$$

with the conditions

$$\phi_{\lambda\lambda}^{(i)} = 0, \quad \psi_{\lambda\lambda} \neq 0, \qquad (i = 1 \dots p;\ \lambda = 1 \dots n), \qquad (7)$$

equations (3) and (6) lead to the following:

$$\begin{aligned} (U_i U_{p+1}) = {} & (\psi_{11} q_1 + \psi_{21} q_2 + \dots + \psi_{n1} q_n) \phi_{11}^{(i)} y_1 \\ & + (\psi_{22} q_2 + \psi_{32} q_3 + \dots + \psi_{n2} q_n)(\phi_{21}^{(i)} y_1 + \phi_{22}^{(i)} y_2) \\ & + \dots\dots\dots\dots \\ & - [\text{same expression with } \phi \text{ and } \psi \text{ interchanged}] \\ = {} & c_{i,p+1,1} U_1 + c_{i,p+1,2} U_2 + \dots + c_{i,p+1,p} U_p, \qquad (8) \\ & (i = 1 \dots p). \end{aligned}$$

Equating coefficients of $y_{\kappa-1} q_\kappa$, we get

$$\begin{aligned} \phi_{\kappa-1,\kappa-1}^{(i)} \psi_{\kappa\kappa} + \phi_{\kappa,\kappa-1}^{(i)} \psi_{\kappa\kappa} - \phi_{\kappa,\kappa-1}^{(i)} \psi_{\kappa-1,\kappa-1} - \phi_{\kappa\kappa}^{(i)} \psi_{\kappa,\kappa-1} \\ = c_{i,p+1,1} \phi_{\kappa,\kappa-1}^{(1)} + c_{i,p+2,2} \phi_{\kappa,\kappa-1}^{(2)} + \dots + c_{i,p+1,p} \phi_{\kappa,\kappa-1}^{(p)}. \qquad (9) \end{aligned}$$

Equations (7) and (9) give the following system of equations:

$$\left.\begin{aligned} & [c_{1,p+1,1} - (\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1})] \phi_{\kappa,\kappa-1}^{(1)} + c_{1,p+1,2} \phi_{\kappa,\kappa-1}^{(2)} + \dots \\ & \qquad + c_{1,p+1,p} \phi_{\kappa,\kappa-1}^{(p)} = 0, \\ & c_{2,p+1,1} \phi_{\kappa,\kappa-1}^{(1)} + [c_{2,p+1,2} - (\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1})] \phi_{\kappa,\kappa-1}^{(2)} + \dots \\ & \qquad + c_{2,p+1,p} \phi_{\kappa,\kappa-1}^{(p)} = 0, \\ & \dots\dots\dots\dots \\ & c_{p,p+1,1} \phi_{\kappa,\kappa-1}^{(1)} + c_{p,p+1,2} \phi_{\kappa,\kappa-1}^{(2)} + \dots \\ & \qquad + [c_{p,p+1,p} - (\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1})] \phi_{\kappa,\kappa-1}^{(p)} = 0, \end{aligned}\right\} (10)$$
$$(\kappa = 2 \dots n).$$

Unless all of the ϕ's in (10) are zero, the determinant of the coefficients vanishes, leading to an equation of degree p for the determination of $\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1}$. Since $\psi_{\kappa\kappa}$ and $\psi_{\kappa-1,\kappa-1}$ are any consecutive roots, we conclude that the root differences are constants, and not more than p of these differences are distinct,

since they are all roots of the same equation of degree p whose coefficients moreover depend merely upon the composition of the group. This result is interesting with regard to the determinateness of the transformation U_{p+1}, if this is to be so determined as to generate with $U_1 \ldots . U_p$ a $p+1$-parameter group of given composition. Since the difference between two consecutive roots, considered as arranged in a certain order, must, according to the theorem, assume one of p constant values, which are fixed beforehand, it follows that after one root ρ_1 has been arbitrarily chosen, only a finite number of values are possible for the other roots.

As in the special case of the last paragraph, this theorem breaks down if the quantities $\phi^{\kappa}_{\lambda, \lambda-1}$ are all zero. Suppose, therefore, that all terms of the form $\phi^{(i)}_{\kappa, \kappa-s} y_{\kappa-s} q_\kappa$ ($s = 0 \ldots . s$; $\varkappa = 1 \ldots . n$; $i = 1 \ldots . p$) vanish. Equating coefficients of $y_{\kappa-s-1} q_\kappa$, we get

$$\begin{aligned}
&\phi^{(i)}_{\kappa-s-1, \kappa-s-1} \psi_{\kappa, \kappa-s-1} + \phi^{(i)}_{\kappa-s, \kappa-s-1} \psi_{\kappa, \kappa-s} + \phi^{(i)}_{\kappa-s+1, \kappa-s-1} \psi_{\kappa, \kappa-s+1} + \ldots . \\
&\quad + \phi^{(i)}_{\kappa-1, \kappa-s-1} \psi_{\kappa, \kappa-1} + \phi^{(i)}_{\kappa, \kappa-s-1} \psi_{\kappa\kappa} \\
&\quad - \phi^{(i)}_{\kappa, \kappa-s-1} \psi_{\kappa-s-1, \kappa-s-1} - \phi^{(i)}_{\kappa, \kappa-s} \psi_{\kappa-s, \kappa-s-1} - \phi^{(i)}_{\kappa, \kappa-s+1} \psi_{\kappa-s+1, \kappa-s-1} - \ldots . \\
&\quad - \phi^{(i)}_{\kappa, \kappa-1} \psi_{\kappa-1, \kappa-s-1} - \phi^{(i)}_{\kappa\kappa} \psi_{\kappa, \kappa-s-1} = c_{1, p+1, 1} \phi^{(1)}_{\kappa-s-1} \\
&\quad + c_{1, p+1, 2} \phi^{(2)}_{\kappa, \kappa-s-1} + \ldots . + c_{1, p+1, p} \phi^{(p)}_{\kappa, \kappa-s-1} .
\end{aligned} \tag{11}$$

All except the two middle terms in the left member of this equation vanish. The following system is therefore obtained :

$$\left.\begin{aligned}
&[c_{1, p+1, 1} - (\psi_{\kappa\kappa} - \psi_{\kappa-s-1, \kappa-s-1})] \phi^{(1)}_{\kappa, \kappa-s-1} + c_{1, p+1, 2} \phi^{(2)}_{\kappa, \kappa-s-1} + \ldots . \\
&\qquad\qquad + c_{1, p+1, p} \phi^{(p)}_{\kappa, \kappa-s-1} = 0, \\
&c_{2, p+1, 1} \phi^{(1)}_{\kappa, \kappa-s-1} + [c_{2, p+1, 2} - (\psi_{\kappa\kappa} - \psi_{\kappa-s-1, \kappa-s-1})] \phi^{(2)}_{\kappa, \kappa-s-1} + \ldots . \\
&\qquad\qquad + c_{2, p+1, p} \phi^{(p)}_{\kappa, \kappa-s-1} = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&c_{p, p+1, 1} \phi^{(1)}_{\kappa, \kappa-s-1} + c_{p, p+1, 2} \phi^{(2)}_{\kappa, \kappa-s-1} + \ldots . \\
&\qquad\qquad + [c_{p, p+1, p} - (\psi_{\kappa\kappa} - \psi_{\kappa-s-1, \kappa-s-1})] \phi^{(p)}_{\kappa, \kappa-s-1} = 0 .
\end{aligned}\right\} \tag{12}$$

This system in $\phi_{\kappa, \kappa-s-1}$ is identical with (10). Unless, therefore, all of the quantities $\phi_{\kappa, \kappa-s-1}$ are zero, we conclude, as before, that the differences $\psi_{\kappa\kappa} - \psi_{\kappa-s-1, \kappa-s-1}$ are the roots of an equation of degree p, whose coefficients are functions of the composition constants. Since s may assume all values from 1 to $\mu - 2$, we conclude that there is at least one pair of roots of the characteristic equation belonging to U_{p+1} whose difference is a constant, otherwise all of the infinitesimal transformations of the p-parameter subgroup would vanish identically.

From the results obtained above follows the additional theorem: *If $U_1 \ldots . U_r$ generate an r-parameter integrable linear or linearoid group which have been put in the form demanded by* (1), *the characteristic equations belonging to $U_1 \ldots . U_{r-1}$ have all their roots equal to zero, provided that for each value of κ from* 1 *to r not all the composition constants $c_{i,\kappa,\kappa-1}$* $(i = 1 \ldots . \kappa - 1)$ *vanish. Under these same conditions the differences between the roots of the characteristic equation belonging to U_r are constants, no more than r of these differences being distinct, provided not all the coefficients in $U_1 \ldots . U_{r-1}$ of the form $\phi^{(\kappa)}_{\lambda,\lambda-1}$* $(\lambda = 1 \ldots . n;$ $\kappa = 1 \ldots . r - 1)$ *vanish. And in any case, provided the first condition holds, there is always one pair of roots whose difference is a constant.*

University of California, *April* 23, 1901.

On a Certain Group of Isomorphisms.

By John Wesley Young.

Let G denote the non-abelian group of order p^m (p an odd prime) which contains operators of order p^{m-1}. It is the object of the present paper to discuss the properties and ultimately (§7) to determine the defining relations of the group of isomorphisms of G. The writer wishes to express his indebtedness to Professor G. A. Miller for helpful suggestions offered during its preparation.

§1.—*Combinatory Laws and Subgroups of G.*

The defining relations of G are*

$$P^{p^{m-1}} = 1, \quad Q^p = 1, \quad Q^{-1}PQ = P^{1+p^{m-2}}.$$

If we denote any operator $P^\alpha Q^\beta$ of G by $[\alpha, \beta]$, the product of any two operators $[\alpha, \beta]$ and $[\gamma, \delta]$ is given by

$$[\alpha, \beta][\gamma, \delta] = [\alpha + \gamma - \beta\gamma p^{m-2}, \quad \beta + \delta]; \tag{I}$$

by repetition we obtain

$$[\alpha, \beta]^n = [n\alpha - \tfrac{1}{2}n(n-1)\alpha\beta p^{m-2}, \quad n\beta]. \tag{II}$$

In these formulæ, the first number in the brackets is to be taken mod p^{m-1}, the second mod p.

All the subgroups in G are abelian. Moreover, G contains just $p + 1$ subgroups of every order p^s, of which p are cyclic (generated by $[p^{m-s-1}, x]$, $x = 0, 1, 2, \ldots, p-1$) and one is abelian of type $(s-1, 1)$, except when

* Burnside, "Theory of Groups of Finite Order" (1897), p. 76.

$s = 1$.* Since, by (II), we have $[\alpha, \beta]^p = [\alpha p, 0]$, the cyclic subgroup of order p^s, generated by $[p^{m-s-1}, 0]$ $(s < m - 1)$, is contained in all the subgroups of order p^{s+1}; it is therefore characteristic. The non-cyclic subgroups are evidently characteristic. Hence, *G contains a characteristic cyclic subgroup of every order p^s, when $1 \leqq s < m - 1$; and a characteristic non-cyclic subgroup of everg order $p^{s'}$, when $1 < s' < m$.*

§2.—*Defining Correspondence of any Holomorphism of G.*

Let the correspondence

$$P \sim P^\alpha Q^\beta, \qquad Q \sim P^{\gamma p^{m-2}} Q^\delta. \qquad [\alpha \not\equiv 0 \bmod p, \beta, \gamma, \delta < p],$$

define an holomorphism† of G. Applying (I), (II) of §1, this assumption leads at once to the correspondence

$$[a, b] \sim [a\alpha + (b\gamma - \tfrac{1}{2} a (a - 1) \alpha\beta) p^{m-2}, \ a\beta + b\delta],$$

where $[a, b]$ is any operator of G. The product of the operators corresponding to any two, $[a, b]$ and $[c, d]$, must be the operator corresponding to $[a, b][c, d] = [a + c - bcp^{m-2}, b + d]$ for all values of a, b, c, d. Calculating the expressions and equating the coefficients of a, b, c, d, we find the necessary and sufficient condition for this equality to be

$$\delta \equiv 1 \mod p.$$

Moreover, it is readily shown that the assumption leads to a one-to-one correspondence. Hence, *the correspondence*

$$C = \begin{Bmatrix} P \sim P^\alpha Q^\beta \quad , \\ Q \sim P^{\gamma p^{m-2}} Q, \end{Bmatrix} \qquad [\alpha \not\equiv 0 \bmod p, \beta < p, \gamma < p$$

defines an holomorphism for any set α, β, γ; and every holomorphism of G is given by C.

Hence, since α may take $p^{m-2}(p-1)$ non-congruent values mod p^{m-1} and β, γ each p non-congruent values mod p, *the order of the group of isomorphism I of G is $p^m (p-1)$.*

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* Burnside, loc. cit., p. 76.

† The word "holomorphism" is used in place of "holoedric isomorphism."

§3.—*Analytic Expression for any Operator in I.—Numbers α, β, γ giving rise to a Subgroup of I.*

Regarding I as a substitution group on the p^m operators of G, any operator S_i of I is completely represented by the expression

$$S_i = \left| \left[a\alpha_i + (b\gamma_i - \tfrac{1}{2} a(a-1)\alpha_i\beta_i)\, p^{m-2},\ a\beta_i + b \right] \right|,$$

where the notation

$$S = \left| [M, N] \right|$$

signifies that S replaces $[a, b]$ by $[M, N]$. For every set $(\alpha_i, \beta_i, \gamma_i)$, S_i represents an operator of I. For a properly chosen system of sets, S_i may be made to represent the operators of any subgroup of I.

Let S_1 in $(\alpha_1, \beta_1, \gamma_1)$ and S_2 in $(\alpha_2, \beta_2, \gamma_2)$ be any two operators of I. Then, if $S_1 S_2$ is an operator of the same kind in $(\alpha_3, \beta_3, \gamma_3)$, by calculating the expression for $S_1 S_2$ in terms of $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ and equating coefficients of a and b, we find

$$2\alpha_3 + \alpha_3\beta_3 p^{m-2} \equiv 2\alpha_1\alpha_2 + \left(2\beta_1\gamma_2 + \alpha_1\alpha_2(\beta_1 + \beta_2)\right) p^{m-2} \bmod p^{\ \,-1}. \quad (\mathrm{A}_1)$$

$$\beta_3 \equiv \beta_1 + \alpha_1\beta_2 \bmod p, \quad (\mathrm{A}_2)$$

$$\gamma_3 \equiv \gamma_2 + \alpha_2\gamma_1 \bmod p. \quad (\mathrm{A}_3)$$

In order that a system of sets $(\alpha_i, \beta_i, \gamma_i)$ may represent a subgroup of I, it is necessary and sufficient that, if from the system any two sets $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ be selected, the set $(\alpha_3, \beta_3, \gamma_3)$ defined by (A_1), (A_2), (A_3) be in the system.

§4.—*Invariant Subgroups of Order p^3 and p^2 in I.*

The correspondence C (§2) shows that the characteristic cyclic subgroup of order p^{m-2} (§1) may correspond to itself in $p^{m-3}(p-1)$ ways; i. e., I contains as a constituent the group of isomorphisms of this cyclic group. The group of isomorphisms of a cyclic group of order p^{m-2} being cyclic, *I contains as a constituent the cyclic group of order $p^{m-3}(p-1)$, and hence an invariant subgroup of order p^3.* Let the latter be denoted by H_3. Since H_3 leaves $\{P^p\}$ unchanged, the operators of H_3 are obtained from the general operator S_i (§3) by restricting α_i to values congruent to unity mod p^{m-2}. Putting $\alpha_i = 1 + \alpha_i' p^{m-2}$, any operator of H_3 is represented by

$${}_3S_i = \left| \left[a + (a\alpha_i' + b\gamma_i - \tfrac{1}{2} a(a-1)\beta_i)\, p^{m-2},\ \ a\beta_i + b \right] \right|,$$

$$[\alpha_i' < p,\ \beta_i < p,\ \gamma_i < p].$$

(It may be noted that the system of sets $(\alpha_i', \beta_i, \gamma_i)$ satisfies the conditions of §3.)

Placing the coefficient of p^{m-2} in ${}_3S_i$ equal to A_i, we way write the n^{th} power of ${}_3S_i$ as follows:

$${}_3S_i^n = |[a + (nA_i + \tfrac{1}{2}n(n-1)\alpha\beta_i\gamma_i)p^{m-2},\ na\beta_i + b]|.$$

Hence $${}_3S_i^p = |[a, b]| = 1;$$

i. e., all the operators of H_3 (except identity) are of order p. Moreover H_3 is non-abelian. For we have

$${}_3S_i \cdot {}_3S_j = |[a + (A_i + A_j + a\beta_i\gamma_j)p^{m-2},\ a(\beta_i + \beta_j) + b]|,$$

which shows that ${}_3S_i$ and ${}_3S_j$ are not in general commutative. Hence, H_3 *is the non-abelian group of order* p^3, *all of whose operators (except identity) are of order* p.*

Hence also I *is non-abelian.*

The characteristic non-cyclic subgroup of order p^{m-1} is made by C to correspond to itself in $p^{m-2}(p-1)$ ways; I therefore contains an invariant subgroup H_2 of order p^2; H_2 being contained in H_3 is abelian of type (1, 1),

§5.—*Invariant Subgroup H_m of Order p^m in I.*

By Sylow's theorem, I contains a single subgroup of order p^m; let it be denoted by H_m. To obtain a general operator ${}_mS_i$ of H_m, we must find a suitable system of sets $(\alpha_i, \beta_i, \gamma_i)$ satisfying (A_1), (A_2), (A_3) of §3.

Restrict α_i to the p^{m-2} values which are congruent to unity mod p; (A_1), (A_2), (A_3) are then equivalent to

$$\begin{aligned}\alpha_3 &\equiv \alpha_1\alpha_2 + \beta_1\gamma_2 p^{m-2} \mod p^{m-1},\\ \beta_3 &\equiv \beta_1 + \beta_2 \mod p,\\ \gamma_3 &\equiv \gamma_1 + \gamma_2 \mod p.\end{aligned}$$

Evidently $(\alpha_3, \beta_3, \gamma_3)$ is in the system for any choice of $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$. Hence the system gives rise to a subgroup of I; this subgroup is of order p^m, and

* Burnside, loc. cit., p. 82.

hence is H_m We have then

$$_mS_i = |[a\alpha_i'' + (b\gamma_i - \tfrac{1}{2}a(a-1)\beta_i)p^{m-2},\ a\beta_i + b]|,$$
$$[\alpha_i'' \equiv 1 \mod p,\ \beta_i < p,\ \gamma_i < p.$$

If $\beta_i = \gamma_i = 0$, the resulting p^{m-2} operators are invariant in H_m, as is readily shown by forming the product $_mS_j'\,{}_mS_i$, where $_mS_j' = |[a\alpha_j'', b]|$. Moreover, H_m contains an invariant operator of order p^{m-2}. For we have

$$_mS_j'^n = |[a\alpha_j''^n, b]|$$

and if $\alpha_j'' = 1 + kp\ (k < p)$, p^{m-2} is the smallest value of n which will satisfy $\alpha_j''^n \equiv 1 \mod p^{m-1}$.

Further, H_m contains no operator of order p^{m-1}. For, we find

$$_mS_i^n = |[a\alpha_i''^n + (nb\gamma_i - \tfrac{1}{2}na(a-1)\beta_i + \tfrac{1}{2}n(n-1)a\beta_i\gamma_i)p^{m-2},\ na\beta_i + b]|$$

and if $n = p^{m-2}$, $_mS_i^n = 1$ for every set $(\alpha_i'', \beta_i, \gamma_i)$.

Hence, H_m *is the non-abelian group of order* p^m *which contains an invariant operator of order* p^{m-2}, *and does not contain any operator of order* p^{m-1}. Its defining relations are :*

$$I_1^{p^{m-2}} = 1,\quad J^p = 1,\quad K^p = 1,\quad K^{-1}JK = JI_1^{p^{m-3}},\quad I_1J = JI_1,\quad I_1K = KI_1.$$

It may be noted that $_mS_j'$ makes every operator of G correspond to its α_j''th power. In a previous paper† it has been shown that all such holomorphisms are invariant in the group of isomorphisms; i. e., the operators $_mS_j'$ are invariant in I, which also follows readily analytically. By another theorem in the same paper, it follows at once that they form a cyclic subgroup H_{m-2} of order p^{m-2} in I; this agrees with the result already obtained.

§6.—*The Quotient-Group* I/H_{m-2}.

H_{m-2} leaves all the subgroups of G fixed. Regarding I/H_{m-2} of order $p^2(p-1)$ as a substitution group on the subgroups of G, it is clear that I/H_{m-2} must contain a transitive constituent T_1 of degree p on the p cyclic subgroups of order p_{m-1}. T_1 is doubly transitive; for the correspondence

$$P \sim P^a,\quad Q \sim Q$$

* Burnside, loc. cit., p. 78.

† Transactions of the American Mathematical Society, Vol. III (1902), p. 189.

leaves $\{P\}$ fixed, while it transforms any of the other cyclic subgroups of order p^{m-1}, $\{PQ\}$, into $\{P^a Q\}$. Non-congruent values of a mod p give different subgroups; by assigning to a any of its $p-1$ non-congruent values mod p, $\{PQ\}$ may be transformed into any one of the remaining $p-1$ subgroups. The order of T_1 cannot be $p^2(p-1)$, since the order must be a divisor of $p\,!$; it must then be $p(p-1)$, since there is no doubly transitive group of degree p and order less than $p(p-1)$. Hence T_1 is a metacyclic group of degree p.

Further, I/H_{m-2} must contain a transitive constituent T_2 of degree p on the p non-characteristic subgroups of G of order p. Reasoning similar to the above shows that T_2 also is a metacyclic group of degree p. Moreover, it is possible to leave the subgroups of order p^{m-1} fixed, and still permute the subgroups of order p according to a group of order p. Hence, I/H_{m-2} *is obtained by establishing a* (p, p)*-isomorphism between two metacyclic groups of degree* p.

§7.—*The Defining Relations of I.*

If we take

$$I_1 = |[a + ap, b]|, \quad J = |[a + bp^{m-2}, b]|,$$
$$K = |[a + \tfrac{1}{2}a(a-1)p^{m-2}, \quad -a + b]|,$$

then I, J, K satisfy the relations defining H_m (§5). For, we have

$$K^{-1}JK = |[a + (a+b)p^{m-2}, b]| = JI_1^{p^{m-3}}$$

and the other five relations may easily be verified.

Corresponding to every operator of order $p-1$ in I/H_{m-2}, there is in I just one operator of order $p-1$. Let

$$I_2 = |[a\alpha, b]|$$

be one of these, where α belongs to exponent $p-1$ mod p^{m-1}. Then, we have

$$I_2^{-1}JI_2 = |[a + \alpha b p^{m-2}, b]| = J^{\alpha}.$$

Also $I_2^{-1}KI_2$ replaces $[a\alpha, b]$ by $[a\alpha + \tfrac{1}{2}a(a-1)\alpha p^{m-2}, \; -a + b]$. This is effected by an operator

$$S_3' = |[a + (a\alpha_3' + \tfrac{1}{2}a(a-1)\beta_3)p^{m-2}, \; -a\beta_3 + b]|,$$

α_3', β_3 being defined by the relations

$$\alpha\beta_3 - 1 \equiv 0, \quad 2\alpha\alpha_3 + \alpha - 1 \equiv 0 \mod p.$$

We find readily

$$S_3' = K^{\beta_3} | [a + a\alpha_3' p^{m-2}, b] | = K^{\beta_3} I_1^{\alpha_3' p^{m-3}}.$$

Placing $I_1 I_2 = I_3$, since I_1 is commutative with I_2, J, K, we may *write the defining relations of I as follows*:

$$I^{p^{m-2}(p-1)} = 1, \quad J^p = 1, \quad K^p = 1, \quad I_3^{-1} J I_3 = J^x, \quad I_3^{p-1} K = K I_3^{p-1},$$
$$I_3^{-1} K I_3 = K^s I_3^{t p^{m-3}(p-1)}, \quad K^{-1} J K = J I_3^{p^{m-3}(p-1)}$$

where x is any number belonging to exponent $p-1$ mod p^{m-1}, and where s, t are defined by the relations

$$\left.\begin{aligned} sx - 1 &\equiv 0, \\ 2xt + x - 1 &\equiv 0, \end{aligned}\right\} \mod p.$$

CORNELL UNIVERSITY.

Isothermal-Conjugate Systems of Lines on Surfaces.

By L. P. Eisenhart.

Bianchi* has given the name *isothermal-conjugate* to a double system of lines, on a surface of positive curvature, which are such that when the surface is referred to them as parametric lines, the second fundamental quadratic form for the surface can be brought to the form

$$\lambda(du^2 + dv^2).$$

We have extended the term so that it refers also to conjugate systems on surfaces of negative curvature; in this case the lines must be such that the second fundamental form can be written

$$\lambda(du^2 - dv^2),$$

when the lines $u = \text{const.}$, $v = \text{const.}$ form an isothermal-conjugate system. It is in this broader sense that we shall treat of such systems.

In the first section we show how these lines can be determined, and by methods similar to those used by Bianchi† in his treatment of isothermal orthogonal systems, we establish theorems similar to those which are well known in the theory of these latter systems. And for surfaces of negative curvature we establish results very similar to those which Bianchi‡ has found for surfaces of positive curvature. A geometrical interpretation of these systems leads to the theorem that the surfaces, obtained by reciprocal radii vectores from surfaces whose lines of curvature are isothermal-conjugate, are of the same kind.

In the second section, we begin the study of surfaces whose lines of curvature form an isothermal-conjugate system, and show that the general problem of determining these surfaces depends upon the integration of a differential equation of the fourth order, very similar to the equation found by Darboux in

* Lezioni, p. 132. † Ib., p. 67. ‡ Bianchi, p. 133.

his study of isothermic surfaces. After showing that the surfaces whose lines of curvature are at the same time isothermal and isothermal-conjugate have an isothermal spherical representation, we pass to the theorem that the sphere, certain surfaces of revolution and minimal surfaces are the only surfaces of constant mean curvature whose lines of curvature are isothermal-conjugate.

By the direct calculation of the coefficients of their second fundamental forms, we show that quadric surfaces and surfaces of revolution have isothermal-conjugate lines of curvature.

By means of a method similar to that used by Willgrod,* we solve the problem of finding all surfaces which, together with their parallels, have isothermal-conjugate lines of curvature We find that the sphere, plane, cyclides of Dupin and surfaces of revolution, are the only surfaces of this kind, and we note further that these surfaces are isothermic.

In addition to the surfaces of constant total curvature and surfaces of revolution, the surfaces whose radii satisfy the conditions

$$\frac{1}{\rho_1\rho_2} = f(U+V), \quad \frac{\rho_2-\rho_1}{\rho_1\rho_2(\rho_1+\rho_2)} = f'(U+V),$$

where f, U, V are bound by a certain relation, have isothermal-conjugate lines of curvature.

The paper closes with a discussion of surfaces with plane lines of curvature in both systems and, at the same time, isothermal-conjugate. Solutions of this problem are furnished by surfaces of revolution, certain moulure surfaces, cyclides of Dupin, minimal surfaces of Bonnet and Enneper and other surfaces whose coordinates are given.

I.

Consider a surface (or region of a surface) of *positive* total curvature. The quadratic differential form

$$f = Ddu^2 + 2D'du\,dv + D''dv^2, \tag{1}$$

which, when equated to zero, gives the imaginary asymptotic directions at the corresponding point, can be decomposed into two conjugate imaginary factors, thus:

$$f = \left(\sqrt{D}du + \frac{D' + i\sqrt{DD''-D'^2}}{\sqrt{D}}dv\right)\left(\sqrt{D}du + \frac{D' - i\sqrt{DD''-D'^2}}{\sqrt{D}}dv\right). \tag{2}$$

* "Ueber Flächen, welche sich durch ihre Krummungslinien in unendlich kleine Quadrate theilen lassen (Dissertation, Göttingen, 1883).

From integral calculus we know that there is an infinity of integrating factors of

$$\sqrt{D}\,du + \frac{D' + i\sqrt{DD'' - D'^2}}{\sqrt{D}}\,dv; \tag{3}$$

let one of them be $\mu + i\nu$, and let $\phi + i\psi$ be the function of which the product of $\mu + i\nu$ and (3) is the exact differential, then

$$(\mu + i\nu)\left(\sqrt{D}\,du + \frac{D' + i\sqrt{DD'' - D'^2}}{\sqrt{D}}\,dv\right) = d\,(\phi + i\psi), \tag{4}$$

and

$$(\mu - i\nu)\left(\sqrt{D}\,du + \frac{D' - i\sqrt{DD'' - D'^2}}{\sqrt{D}}\,dv\right) = d\,(\phi - i\psi). \tag{5}$$

If, now, the surface is referred to the system of lines $\phi = \text{const.}$, $\psi = \text{const.}$, where ϕ and ψ are the functions determined by equations (4) and (5), the form f becomes

$$f = \frac{1}{\mu^2 + \nu^3}\,(d\phi^2 + d\psi^2). \tag{6}$$

Hence, the system of lines $\phi = \text{const.}$, $\psi = \text{const.}$ is isothermal-conjugate. We remark that its determination depends upon the integration of the differential equation

$$\sqrt{D}\,du + \frac{D' + i\sqrt{DD'' - D'^2}}{\sqrt{D}}\,dv = 0. \tag{7}$$

Returning to the general coordinates u and v, we form the Beltrami differential parameters with respect to the form (1); they are

$$\Delta\theta = \frac{D\left(\frac{\partial\theta}{\partial v}\right)^2 - 2D'\frac{\partial\theta}{\partial v}\frac{\partial\theta}{\partial u} + D''\left(\frac{\partial\theta}{\partial u}\right)^2}{DD'' - D'^2}, \tag{8}$$

$$\Delta(\phi, \theta) = \frac{D\frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial v} - D'\left(\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial v} + \frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial u}\right) + D''\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial u}}{DD'' - D'^2}, \tag{9}$$

$$\Delta_2\theta = \frac{1}{\sqrt{DD'' - D'^2}}\left[\frac{\partial}{\partial u}\left(\frac{D''\frac{\partial\theta}{\partial u} - D'\frac{\partial\theta}{\partial v}}{\sqrt{DD'' - D'^2}}\right) + \frac{\partial}{\partial v}\left(\frac{D\frac{\partial\theta}{\partial v} - D'\frac{\partial\theta}{\partial u}}{\sqrt{DD'' - D'^2}}\right)\right]. \tag{10}$$

Let the surface be referred to the new system of lines $\phi =$ const., $\psi =$ const.; then

$$f = D_1 \, d\phi^2 + 2D_1' \, d\phi \, d\psi + D_1'' \, d\psi^2, \tag{11}$$

or, this may be written,

$$f = \frac{\Delta\psi \, d\phi^2 - 2\Delta(\phi, \psi) \, d\phi \, d\psi + \Delta\phi \, d\psi^2}{\Delta\phi \, . \, \Delta\psi - \Delta^2(\phi, \psi)}, \tag{12}$$

where, in consequence of their invariant character, the differential parameters may be formed either with respect to the form (1) or (11). Assuming that they are as given by (8), (9), (10), we remark that if the new variables ϕ and ψ are chosen in such a way that

$$\Delta\phi = \Delta\psi, \quad \Delta(\phi, \psi) = 0, \tag{13}$$

the curves $\phi =$ const., $\psi =$ const. will form an isothermal-conjugate system, and conversely.

Let $\phi(u, v)$ be a particular solution of the partial differential equation

$$\Delta_2 \theta = 0, \tag{14}$$

that is,

$$\frac{\partial}{\partial u}\left(\frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{\sqrt{DD'' - D'^2}}\right) + \frac{\partial}{\partial v}\left(\frac{D \frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u}}{\sqrt{DD'' - D'^2}}\right) = 0. \tag{15}$$

In consequence of this we see that

$$-\frac{D \frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u}}{\sqrt{DD'' - D'^2}} \, du + \frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{\sqrt{DD'' - D'^2}} \, dv$$

is an exact differential. Denote the latter by $d\psi$, then

$$\frac{\partial \psi}{\partial u} = -\frac{D \frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u}}{\sqrt{DD'' - D'^2}}, \quad \frac{\partial \psi}{\partial v} = \frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{\sqrt{DD'' - D'^2}}. \tag{16}$$

Solving for $\frac{\partial \phi}{\partial u}$, $\frac{\partial \phi}{\partial v}$, we have

$$\frac{\partial \phi}{\partial u} = \frac{D \frac{\partial \psi}{\partial v} - D' \frac{\partial \psi}{\partial u}}{\sqrt{DD'' - D'^2}} \quad \frac{\partial \phi}{\partial v} = -\frac{D'' \frac{\partial \psi}{\partial u} - D' \frac{\partial \psi}{\partial v}}{\sqrt{DD'' - D'^2}}. \tag{17}$$

From this it follows that the function ψ, determined by quadratures from (16),

is also a solution of equation (14). Hence, when one solution of this equation is known, another is given by quadratures. Two such solutions will be called *conjugate to one another.* It follows at once from (16) and (17) that ϕ and ψ satisfy the conditions

$$\Delta\phi = \Delta\psi, \quad \Delta(\phi, \psi) = 0.$$

Recalling the preceding results, we have that, *if ϕ and ψ are conjugate solutions of the equation*

$$\Delta_2\theta = 0,$$

the curves $\phi =$ const., $\psi =$ const. form an isothermal-conjugate system.

If the u and v lines form an isothermal-conjugate system, equations (17) reduce to

$$\frac{\partial\phi}{\partial u} = \frac{\partial\psi}{\partial v}, \quad \frac{\partial\phi}{\partial v} = -\frac{\partial\psi}{\partial u},$$

that is, $\phi + i\psi$ is a function of $u + iv$. Since the form (6) is not changed when ψ is replaced by $-\psi$, we have the theorem:

When an isothermal-conjugate system (ϕ, ψ) for a surface of positive curvature is known, every other isothermal-conjugate system (ϕ', ψ') can be obtained by putting

$$\phi' + i\psi' = F(\phi \pm i\psi),$$

where F is an arbitrary function.

Suppose that for a given conjugate system of lines $u =$ const., $v =$ const., we have

$$\frac{D}{D''} = \frac{U}{V}, \tag{18}$$

where U is a function of u alone and V is a function of v alone. Equation (18) can be replaced by

$$D = \lambda U, \quad D'' = \lambda V,$$

where λ is a function of u and v. We have now

$$f = \lambda(U du^2 + V dv^2). \tag{19}$$

If we introduce new variables u_1 and v_1, defined by

$$u_1 = \int \sqrt{U} du, \quad v_1 = \int \sqrt{V} dv, \tag{20}$$

the above form becomes

$$f = \lambda(du_1^2 + dv_1^2). \tag{19'}$$

Hence, the lines $u_1 = \text{const.}$, $v_1 = \text{const.}$ form an isothermal-conjugate system. But, from (20), it follows that this is the same system as $u = \text{const.}$, $v = \text{const.}$ Therefore, when the condition (18) is satisfied by a conjugate system of lines, the system is isothermal-conjugate, and conversely.

From the above it follows that if the curves $\phi = \text{const.}$ and their conjugates form an isothermal-conjugate system, it is possible to find a function F such that $F(\phi)$ is a particular solution of the equation

$$\Delta_2 \theta = 0.$$

Conversely, if $F(\phi)$ satisfies this equation, the curves $\phi = \text{const.}$ and their conjugate trajectories form an isothermal-conjugate system.

From (10) we have that

$$\Delta_2 [F(\phi)] = F'(\phi)\,\Delta_2\phi + F''(\phi)\,\Delta\phi = 0,$$

where accents denote differentiation with respect to ϕ. Hence,

$$\frac{\Delta_2 \phi}{\Delta\phi} = -\frac{F''(\phi)}{F'(\phi)}. \tag{21}$$

Since the right-hand member is a function of ϕ alone, the left-hand member is a function of ϕ alone. Conversely, if $\dfrac{\Delta_2\phi}{\Delta\phi}$ is a function of ϕ alone, $F(\phi)$ can be so determined that

$$\Delta_2 [F(\phi)] = 0.$$

We have then the following theorem: *The necessary and sufficient condition that a family of lines $\phi = \text{const.}$ and their conjugates form an isothermal-conjugate system, is that the ratio of the second and first differential parameters, formed with respect to the second fundamental quadratic form of the surface, is a function of ϕ alone.*

The conjugate trajectories of the curves $\phi = \text{const.}$ are found by the quadratures

$$\frac{\partial\psi}{\partial u} = -F'(\phi)\,\frac{D\dfrac{\partial\phi}{\partial v} - D'\dfrac{\partial\phi}{\partial u}}{\sqrt{DD'' - D'^2}}, \quad \frac{\partial\psi}{\partial v} = F'(\phi)\,\frac{D''\dfrac{\partial\phi}{\partial u} - D'\dfrac{\partial\phi}{\partial v}}{\sqrt{DD'' - D'^2}}, \tag{22}$$

where

$$F'(\phi) = e^{-\int \frac{\Delta_2\phi}{\Delta\phi}\,d\phi}.$$

It follows from this that *if the lines $\phi = \text{const.}$ belong to a double isothermal-conju-*

gate system, their conjugate trajectories are given by the differential equation

$$\frac{D\dfrac{\partial\phi}{\partial v} - D'\dfrac{\partial\phi}{\partial u}}{\sqrt{DD''-D'^2}}\,du - \frac{D''\dfrac{\partial\phi}{\partial u} - D'\dfrac{\partial\phi}{\partial v}}{\sqrt{DD''-D'^2}}\,dv = 0,$$

where $e^{-\int\frac{\Delta_2\phi}{\Delta\phi}d\phi}$ *is an integrating factor.*

Suppose now that the family $\phi = \text{const.}$ of a double isothermal-conjugate system of lines is given by a differential equation of the first order,

$$M\,du + N\,dv = 0;$$

then the other family of lines is given by

$$\frac{DN - D'M}{\sqrt{DD''-D'^2}}\,du + \frac{D'N - D''M}{\sqrt{DD''-D'^2}}\,dv = 0;$$

From what precedes we know that these two equations have the same integrating factor; let it be denoted by μ and for the sake of brevity write

$$M_1 = \frac{DN - D'M}{\sqrt{DD''-D'^2}}, \quad N_1 = \frac{D'N - D''M}{\sqrt{DD''-D'^2}}.$$

Then we find

$$\frac{\partial \log \mu}{\partial u} = \frac{M_1\left(\dfrac{\partial N}{\partial u} - \dfrac{\partial M}{\partial v}\right) - M\left(\dfrac{\partial N_1}{\partial u} - \dfrac{\partial M_1}{\partial v}\right)}{MN_1 - M_1N} \equiv A,$$

$$\frac{\partial \log \mu}{\partial v} = \frac{N_1\left(\dfrac{\partial N}{\partial u} - \dfrac{\partial M}{\partial v}\right) - N\left(\dfrac{\partial N_1}{\partial u} - \dfrac{\partial M_1}{\partial v}\right)}{MN_1 - M_1N} \equiv B,$$

from which μ can be obtained by quadratures. Hence, we have the following theorem analogous to that due to Lie concerning isothermal systems:

When one family of a double isothermal-conjugate system of lines on a surface of positive curvature is given by a differential equation of the first order, viz.,

$$M\,du + N\,dv = 0,$$

the equation of the lines in finite terms can be obtained by quadratures.

From equations (4) and (5) it follows that when $\phi = \text{const.}$, $\psi = \text{const.}$ form an isothermal-conjugate system, the asymptotic lines are given by

$$\phi + i\psi = \text{const.}, \quad \phi - i\psi = \text{const.}$$

Consider now surfaces whose total curvature is negative. The problem of finding an isothermal-conjugate system of lines on such a surface is the same as that of finding the asymptotic lines. For if u, v are the parameters referring to the asymptotic lines, the quadratic form f is simply

$$f = D' \, du \, dv .$$

Let now ϕ and ψ be two functions of u and v defined by the equations

$$\phi = f_1(u) + f_2(v), \quad \psi = f_1(u) - f_2(v), \tag{23}$$

where f_1 and f_2 are arbitrary functions. When the surface is referred to the system of lines $\phi = \text{const.}$, $\psi = \text{const.}$, we have

$$f = \lambda (d\phi^2 - d\psi^2),$$

where λ is a function of ϕ and ψ. Hence, *if the lines* $u = const.$, $v = const.$ *are asymptotic lines, an isothermal-conjugate system of lines is given by*

$$f_1(u) + f_2(v) = \text{const.}, \quad f_1(u) - f_2(v) = \text{const.}$$

Conversely, if $\phi = const.$, $\psi = const.$ *form an isothermal-conjugate system, then the asymptotic lines are given by*

$$F_1(\phi + \psi) = \text{const.}, \quad F_2(\phi - \psi) = \text{const.}$$

Let the surface be referred to an isothermal-conjugate system (u, v); then

$$D = -D'', \quad D' = 0 .$$

Let X, Y, Z denote the direction cosines of the normal to the surface, and write

$$\mathfrak{E} = \Sigma \left(\frac{\partial X}{\partial u}\right)^2, \quad \mathfrak{F} = \Sigma \frac{\partial X}{\partial u} \frac{\partial X}{\partial v}, \quad \mathfrak{G} = \Sigma \left(\frac{\partial X}{\partial v}\right)^2,$$

then
$$d\sigma^2 = \mathfrak{E} \, du^2 + 2\mathfrak{F} \, du \, dv + \mathfrak{G} \, dv^2$$

is the square of the linear element of the spherical representation of the surface. Denoting by $\begin{Bmatrix} rs \\ t \end{Bmatrix}'$ the Christoffel symbols formed with respect to this quadratic form, and writing

$$\rho = \frac{D}{\sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}} = \frac{-D''}{\sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}},$$

we have for the Codazzi equations, referring to the spherical representation of the surface,*

$$\frac{\partial \log \rho}{\partial v} = \left\{ {11 \atop 2} \right\}' - \left\{ {22 \atop 2} \right\}', \quad \frac{\partial \log \rho}{\partial u} = \left\{ {22 \atop 1} \right\}' - \left\{ {11 \atop 1} \right\}', \tag{24}$$

and if (x, y, z) are the cartesian coordinates of a point on the surface, their derivatives with respect to u and v are given by

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= \frac{\rho}{\sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}} \left(-\mathcal{G} \frac{\partial X}{\partial u} + \mathcal{F} \frac{\partial X}{\partial v} \right), \\ \frac{\partial x}{\partial v} &= -\frac{\rho}{\sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}} \left(\mathcal{F} \frac{\partial X}{\partial u} - \mathcal{E} \frac{\partial X}{\partial v} \right), \end{aligned}\right\} \tag{25}$$

and analogous expressions in y and z.

From (24) it follows that *the necessary and sufficient condition that a system of u, v lines on a sphere be the spherical representation of an isothermal-conjugate system of lines on a surface of negative curvature is expressed by*

$$\frac{\partial}{\partial u} \left(\left\{ {11 \atop 2} \right\}' - \left\{ {22 \atop 2} \right\}' \right) = \frac{\partial}{\partial v} \left(\left\{ {22 \atop 1} \right\}' - \left\{ {11 \atop 1} \right\}' \right). \tag{26}$$

If we put

$$\sqrt{\rho}\, X = \xi, \quad \sqrt{\rho}\, Y = \eta, \quad \sqrt{\rho}\, Z = \zeta, \tag{27}$$

equations (25) may be replaced by

$$\frac{\partial x}{\partial u} = + \begin{vmatrix} \eta, & \zeta \\ \frac{\partial \eta}{\partial v}, & \frac{\partial \zeta}{\partial v} \end{vmatrix}, \quad \frac{\partial x}{\partial v} = + \begin{vmatrix} \eta, & \zeta \\ \frac{\partial \eta}{\partial u}, & \frac{\partial \zeta}{\partial u} \end{vmatrix}, \tag{28}$$

and analogous ones in y and z. It is readily shown that X, Y, Z are particular solutions of the equation

$$\frac{\partial^2 \phi}{\partial u^2} - \frac{\partial^2 \phi}{\partial v^2} + \frac{\partial \log \rho}{\partial u} \frac{\partial \phi}{\partial u} - \frac{\partial \log \rho}{\partial v} \frac{\partial \phi}{\partial v} + (\mathcal{E} - \mathcal{G})\phi = 0.$$

Effecting the transformation

$$\sqrt{\rho}\, \phi = \theta,$$

we can bring this equation to the form

$$\frac{\partial^2 \theta}{\partial u^2} - \frac{\partial^2 \theta}{\partial v^2} = M\theta, \quad M = \frac{1}{\sqrt{\rho}} \left(\frac{\partial^2 \sqrt{\rho}}{\partial u^2} - \frac{\partial^2 \sqrt{\rho}}{\partial v^2} \right) - (\mathcal{E} - \mathcal{G}); \tag{29}$$

* Bianchi, Lezioni, p. 131.

from (27) it follows that ξ, η, ζ are particular solutions of this equation. Conversely, if we take any three particular solutions ξ, η, ζ of an equation of the form (29) where M is any function of u and v, the surface, whose cartesian coordinates are determined from these solutions by means of (28), will have the u, v lines for an isothermal-conjugate system.

Recalling the expressions of Gauss for the second derivatives of the cartesian coordinates of a point on a surface,* we find that when the surface is one of negative curvature and is referred to an isothermal-conjugate system of lines, these coordinates must be simultaneous solutions of the equations

$$\left.\begin{aligned} \frac{\partial^2\theta}{\partial u\,\partial v} &= \left\{\begin{matrix}12\\1\end{matrix}\right\}\frac{\partial\theta}{\partial u} + \left\{\begin{matrix}12\\2\end{matrix}\right\}\frac{\partial\theta}{\partial v}, \\ \frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} &= \left[\left\{\begin{matrix}11\\1\end{matrix}\right\} + \left\{\begin{matrix}22\\1\end{matrix}\right\}\right]\frac{\partial\theta}{\partial u} + \left[\left\{\begin{matrix}11\\2\end{matrix}\right\} + \left\{\begin{matrix}22\\2\end{matrix}\right\}\right]\frac{\partial\theta}{\partial v}, \end{aligned}\right\} \tag{30}$$

where the Christoffel symbols $\left\{\begin{matrix}rs\\t\end{matrix}\right\}$ are formed with respect to the square of the linear element of the surface.

Conversely, if x, y, z are three linearly independent solutions of a completely integrable system of the form

$$\left.\begin{aligned} \frac{\partial^2\theta}{\partial u\,\partial v} &= a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \\ \frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} &= \alpha\frac{\partial\theta}{\partial u} + \beta\frac{\partial\theta}{\partial v}, \end{aligned}\right\} \tag{31}$$

then x, y, z are the cartesian coordinates of a point on a surface of negative curvature referred to an isothermal-conjugate system of lines.

The results of the preceding sections are similar to those deduced by Bianchi† for surfaces of positive curvature referred to an isothermal-conjugate system; the various equations and conditions differ at most in sign.

It is readily seen that the plane is the only surface of zero curvature which admits isothermal-conjugate lines as defined. For, in this case,

$$DD'' - D'^2 = 0,$$

and hence only when

$$D = D' = D'' = 0,$$

can the above condition and that for isothermal-conjugate lines be satisfied simultaneously.

* Bianchi, p. 88. † Lezioni, pp. 133, 134.

If the distance from the point $(u + du, v + dv)$ on a surface to the tangent plane to the surface at the point (u, v) be denoted by p, it can be shown that

$$p = \tfrac{1}{2}(D\,du^2 + 2D'\,du\,dv + D''\,dv^2) + \ldots\ldots,$$

where the unwritten terms are of a higher order than the second. Neglecting the latter, we find that the distance from $(u+du, dv)$ to this plane is $\frac{1}{2}Ddu^2$ and from $(u, v+dv)$ is $\frac{1}{2}D''dv^2$. If, then, we take $du = dv$, we find that when u and v refer to an isothermal-conjugate system, these distances are equal; moreover, they are measured in the same direction at points of positive curvature and in opposite directions at points of negative curvature.

Consider now, in connection with a surface S, whose u and v lines form an isothermal-conjugate system, the surface S_1 which corresponds to S by reciprocal radii vectores. Then, if x_1, y_1, z_1 denote the cartesian coordinates of a point on S_1, we have

$$x = \frac{\kappa^2 x_1}{x_1^2 + y_1^2 + z_1^2}, \quad y = \frac{\kappa^2 y_1}{x_1^2 + y_1^2 + z_1^2}, \quad z = \frac{\kappa^2 z_1}{x_1^2 + y_1^2 + z_1^2}. \tag{32}$$

From the geometrical interpretation of isothermal-conjugate lines, as given above, and from the well-known fact that the transformation (32) changes a plane into a plane, a conjugate system of lines on S into a conjugate system on S_1, preserves angles and the ratios of infinitely small lengths, it follows that on S_1 the system of u and v lines form an isothermal-conjugate system. And since Darboux has shown that the lines of curvature on S_1 correspond to the lines of curvature on S, we have the theorem:

The transforms by reciprocal radii vectores of surfaces whose lines of curvature form an isothermal-conjugate system are also surfaces of this kind.

II.

Consider a surface S referred to a general system of curvilinear coordinates (u, v); the square of its linear element takes the form

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{1}$$

The condition that the coordinate lines be the lines of curvature is

$$F = 0, \quad D' = 0,$$

and that these lines form an isothermal-conjugate system is, with proper choice of parameters,

$$D = \pm D'',$$

according as the surface is of positive or negative curvature.

Combining with the results found in the previous section with respect to surfaces of negative curvature, the similar results found by Bianchi for surfaces of positive curvature, we have that, in order that the lines of curvature of a surface may form an isothermal-conjugate system, it is necessary that the cartesian coordinates of its points satisfy simultaneously the two equations

$$\left.\begin{aligned} \frac{\partial^2\theta}{\partial u\,\partial v} &= \frac{\partial \log \sqrt{E}}{\partial v}\,\frac{\partial\theta}{\partial u} + \frac{\partial \log\sqrt{G}}{\partial u}\,\frac{\partial\theta}{\partial v}, \\ \frac{\partial^2\theta}{\partial u^2} \mp \frac{\partial^2\theta}{\partial v^2} &= \frac{1}{2E}\left(\frac{\partial E}{\partial u} \pm \frac{\partial G}{\partial u}\right)\frac{\partial\theta}{\partial u} \mp \left(\frac{\partial E}{\partial v} \pm \frac{\partial G}{\partial v}\right)\frac{\partial\theta}{\partial v}, \end{aligned}\right\} \tag{2}$$

where the upper sign holds for surfaces of positive curvature, and the lower for surfaces of negative curvature. Conversely, if two simultaneous equations of the form

$$\left.\begin{aligned} \frac{\partial^2\theta}{\partial u\,\partial v} &= a\,\frac{\partial\theta}{\partial u} + b\,\frac{\partial\theta}{\partial v}, \\ \frac{\partial^2\theta}{\partial u^2} \mp \frac{\partial^2\theta}{\partial v^2} &= \alpha\,\frac{\partial\theta}{\partial u} + \beta\,\frac{\partial\theta}{\partial v}, \end{aligned}\right\} \tag{3}$$

constitute a completely integrable system, and x, y, z are linearly independent solutions, satisfying the conditions

$$\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\,\frac{\partial z}{\partial v} = 0,$$

then the locus of the point (x, y, z) is a surface of positive or negative curvature (according to the signs in the second equation (3)), whose lines of curvature form an isothermal-conjugate system.

It is evident that the problem of finding all the surfaces whose lines of curvature form an isothermal-conjugate system is a very difficult one, especially from the preceding point of view. We shall, however, by different methods find quite a number of such surfaces, but before proceeding to this, we will show that the solution of our problem depends upon the integration of a differential equation of the fourth order, similar to that found by Darboux for isothermic surfaces.

To this end we consider the surface referred to the system of tangential coordinates, imagined by Bonnet, for which the equation of the tangent plane takes the form

$$(\alpha + \beta)\,x + i\,(\beta - \alpha)\,y + (\alpha\beta - 1)\,z + \xi = 0. \tag{4}$$

Then the lines of curvature are given by

$$r\,d\alpha^2 - t\,d\beta^2 = 0,^* \tag{5}$$

and the z coordinate of the point of contact has the expression

$$z = \frac{\xi - p\alpha - q\beta}{1 + \alpha\beta}, \tag{6}$$

where we write

$$p = \frac{\partial \xi}{\partial \alpha}, \quad q = \frac{\partial \xi}{\partial \beta}, \quad r = \frac{\partial^2 \xi}{\partial \alpha^2}, \quad s = \frac{\partial^2 \xi}{\partial \alpha\, \partial \beta}, \quad t = \frac{\partial^2 \xi}{\partial \beta^2}. \tag{7}$$

The second fundamental form is

$$f = \{(1 + \alpha\beta)(r\,d\alpha^2 + 2s\,d\alpha\,d\beta + t\,d\beta^2) + 2d\alpha\,d\beta\,(\xi - p\alpha - q\beta)\}\varkappa, \tag{8}$$

where $\varkappa$ is a function of α and β whose form is not essential.

If the parameters of the lines of curvature are u, v, then

$$du = \lambda(\sqrt{r}\,d\alpha - \sqrt{t}\,d\beta), \quad dv = \mu(\sqrt{r}\,d\alpha + \sqrt{t}\,d\beta), \tag{9}$$

where λ and μ are such that the expressions on the right are exact differentials. If the lines of curvature of the surface S form an isothermal-conjugate system,

$$f = D\,(du^2 + dv^2), \tag{10}$$

where S is a surface of positive curvature, as we will assume it is for the present. Replace du, dv in (10) by their expressions (9) and compare with (8); this gives

$$\left.\begin{array}{l} Dr\,(\lambda^2 + \mu^2) = \varkappa\,(1 + \alpha\beta)\,r, \quad Dt\,(\lambda^2 + \mu^2) = \varkappa\,(1 + \alpha\beta)\,t, \\ D\sqrt{rt}\,(\mu^2 - \lambda^2) = \varkappa\,[s\,(1 + \alpha\beta) + \xi - p\alpha - q\beta] = \varkappa\,(1 + \alpha\beta)(s + z). \end{array}\right\} \tag{11}$$

Since the first two of these conditions are the same, we have two equations in λ^2 and μ^2; solving, we get

$$2D\lambda^2 = -\frac{s + z - \sqrt{rt}}{\sqrt{rt}}\,(1 + \alpha\beta)\,\varkappa, \quad 2D\mu^2 = \frac{s + z + \sqrt{rt}}{\sqrt{rt}}\,(1 + \alpha\beta)\,\varkappa. \tag{12}$$

Put

$$2D = \frac{(s + z)^2 - rt}{\theta^2\sqrt{rt}}\,(1 + \alpha\beta)\,\varkappa, \tag{13}$$

then

$$\lambda = \frac{i\theta}{(s + z + \sqrt{rt})^{\frac{1}{2}}}, \quad \mu = \frac{\theta}{(s + z - \sqrt{rt})^{\frac{1}{2}}}.$$

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* Darboux, Leçons, I, p. 246.

Hence θ must be such that

$$\theta \frac{\sqrt{r}\,d\alpha - \sqrt{t}\,d\beta}{(s+z+\sqrt{rt})^{\frac{1}{2}}}, \qquad \theta \frac{\sqrt{r}\,d\alpha + \sqrt{t}\,d\beta}{(s+z-\sqrt{rt})^{\frac{1}{2}}}$$

are exact differentials. The condition of integrability of the first is

$$\sqrt{r}\frac{\partial \log\theta}{\partial\beta} + \sqrt{t}\frac{\partial\log\theta}{\partial\alpha} + \frac{\partial\sqrt{r}}{\partial\beta} + \frac{\partial\sqrt{t}}{\partial\alpha} - \sqrt{r}\frac{\partial}{\partial\beta}\log(z+s+\sqrt{rt})^{\frac{1}{2}}$$
$$- \sqrt{t}\frac{\partial}{\partial\alpha}\log(z+s+\sqrt{rt})^{\frac{1}{2}} = 0.$$

The condition for the second is gotten from this by replacing $\sqrt{t}$ by $-\sqrt{t}$. From these two equations of condition we obtain the following:

$$2d\log\frac{\theta}{[(z+s)^2-rt]^{\frac{1}{4}}} = -\frac{\partial\log t}{\partial\alpha}d\alpha - \frac{\partial\log r}{\partial\beta}d\beta$$
$$+\sqrt{\frac{r}{t}}\frac{\partial}{\partial\beta}\log\left(\frac{z+s+\sqrt{rt}}{z+s-\sqrt{rt}}\right)^{\frac{1}{2}}d\alpha$$
$$+\sqrt{\frac{t}{r}}\frac{\partial}{\partial\alpha}\log\left(\frac{s+z+\sqrt{rt}}{s+z-\sqrt{rt}}\right)^{\frac{1}{2}}d\beta.$$

Expressing the condition that the right-hand member is an exact differential, we find the following equation of the fourth order which ξ must satisfy in order that the lines of curvature of S may form an isothermal-conjugate system:

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}\log\frac{r}{t} - \frac{\partial}{\partial\alpha}\left[\sqrt{\frac{t}{r}}\frac{\partial}{\partial\alpha}\log\left(\frac{s+z+\sqrt{rt}}{s+z-\sqrt{rt}}\right)^{\frac{1}{2}}\right]$$
$$+\frac{\partial}{\partial\beta}\left[\sqrt{\frac{r}{t}}\frac{\partial}{\partial\beta}\log\left(\frac{s+z+\sqrt{rt}}{s+z-\sqrt{rt}}\right)^{\frac{1}{2}}\right] = 0. \quad (14)$$

When S is a surface of negative curvature, we find, by methods similar to the preceding, that for the lines of curvature to form an isothormal-conjugate system we must have

$$Dr(\lambda^2-\mu^2) = (1+\alpha\beta)r, \quad Dt(\lambda^2-\mu^2) = (1+\alpha\beta)t,$$
$$-D\sqrt{rt}(\mu^2+\lambda^2) = (1+\alpha\beta)(s+z),$$

from which, after making use of (13), we have

$$\lambda = \frac{i\theta}{(s+z+\sqrt{rt})^{\frac{1}{2}}}, \quad \mu = \frac{i\theta}{(s+z-\sqrt{rt})^{\frac{1}{2}}}.$$

Proceeding as before, we are led to the same equation (14). Hence, when the

function ξ for a surface of positive or negative curvature satisfies equation (14), the lines of curvature, which can be determined by quadratures by means of (9), form an isothermal-conjugate system.

The expression
$$\frac{s + z + \sqrt{rt}}{s + z - \sqrt{rt}}$$
is the ratio of the two radii of principal curvature of the surface.* Hence, since for a minimal surface,†
$$\frac{r}{t} = \frac{\phi(\alpha)}{f(\beta)},$$
it follows that the above equation is satisfied by the function ξ corresponding to minimal surfaces.

This result can be obtained also as follows: Denote by ρ_1 and ρ_2 the principal radii of curvature at a point, and let the surface be referred to its lines of curvature; then
$$\frac{1}{\rho_1} = \frac{D}{E}, \quad \frac{1}{\rho_2} = \frac{D''}{G}.$$
If, now, S is a minimal surface, it is isothermic, and hence the parameters of the lines of curvature can be so chosen that $E = G$, and hence
$$D = -D''.$$
In a similar manner consider a sphere. Since an orthogonal system upon a sphere is at the same time conjugate and since the two fundamental forms are in constant ratio, every isothermal system upon the sphere is an isothermal-conjugate system and consequently the sphere can be looked upon in an infinity of ways as a surface with its lines of curvature forming an isothermal-conjugate system. Monge‡ has shown that the sphere is the only real surface whose principal radii of curvature are equal and of the same sign, hence minimal surfaces and sphere are the only real surfaces to be investigated by this method.

Recalling the notation of the previous section, we have the following relations, when a surface is referred to its lines of curvature :§
$$\mathfrak{E} = \frac{D^2}{E}, \quad \mathfrak{G} = \frac{D''^2}{G}.$$

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* Darboux, Leçons, I, p. 246.

† Ib., p. 298.

‡ "Application de l'analyse à la géométrie," 5me ed., pp. 196-211.

§ Bianchi, Lezioni, p. 131.

From this it follows that when S is an isothermic surface and the spherical representation of its lines of curvature is also isothermal, then the lines of curvature of S form an isothermal-conjugate system. By means of this we shall be able to determine what surfaces of constant mean curvature have isothermal-conjugate lines of curvature. Bonnet* was the first to show that such surfaces are isothermic.

Willgrod† shows that for surfaces with constant mean curvature different from zero, the parameters of the lines of curvature can be so chosen that the following relations hold:

$$\mathfrak{E}\rho_1^2 = \mathfrak{G}\rho_2^2 = \frac{\rho_1 + \rho_2}{\rho_1 - \rho_2}.$$

From this it follows that for the spherical representation of S to be isothermal we must have

$$\frac{\rho_1}{\rho_2} = \frac{U}{V},$$

where U is a function of u alone and V is a function of v alone. But ρ_1 and ρ also satisfy the condition

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{c},$$

where c is a constant. From these two equations we have

$$\rho_1 = \frac{c(U + V)}{V}, \quad \rho_2 = \frac{c(U + V)}{U}.$$

When these expressions are substituted in the above equation, we get for $\mathfrak{E}$ and $\mathfrak{G}$

$$\mathfrak{E} = \frac{V^2}{c^2(U^2 - V^2)}, \quad \mathfrak{G} = \frac{U^2}{c^2(U^2 - V^2)}.$$

From these forms it is seen that in this case the orthogonal system upon the sphere is such that by a suitable choice of parameters the linear element can be given the form

$$d\sigma^2 = \frac{1}{(U_1 + V_1)}(du_1^2 + dv_1^2).$$

* "Mémoire sur la theorie des surfaces applicables sur une surface donnée" (Journal de l'Ècole Polytechnique, XLII, Cahier, p. 77, 1867).

† Dissertation p. 30.

In order to determine the form of the functions U_1 and V_1, we note that the Gauss equation for the sphere referred to any isothermal orthogonal system, namely:*

$$\frac{\partial^2 \log \lambda}{du^2} + \frac{\partial^2 \log \lambda}{\partial v^2} + 2\lambda = 0,$$

where

$$\lambda = \mathfrak{E} = \mathfrak{G}$$

takes in this case the form

$$(U_1'' + V_1'' - 2)(U_1 + V_1) - U_1'^2 - V_1'^2 = 0,$$

where the accents denote differentiation with respect to u_1 and v_1 respectively. For the present we exclude the case where either U_1 or V_1 is a constant and we put

$$U_2 = U_1'^2, \quad V_2 = V_1'^2,$$

and take U_1 and V_1 for the independent variables. If we make this substitution and denote now by accents differentiation with respect to these new variables, the above equation becomes

$$(U_2' + V_2' - 4)(U_1 + V_1) - U_2 - V_2 = 0.$$

Differentiating this equation with respect to U_1 and then with respect to V_1, we are brought to the equation

$$U_2'' + V_2'' = 0.$$

From this it follows that U_2'' and V_2'' are constants differing only in sign. Hence

$$U_2 = \alpha U_1^2 + 2\beta U_1 + \gamma, \quad V_2 = -\alpha V_1^2 + 2b V_1 + c,$$

where $\alpha, \beta, \gamma, b, c$ are constants. When these values are substituted in the above equation, it becomes

$$U_1(2 + \beta - b) + V_1(2 - \beta + b) + \gamma + c = 0.$$

From the form of this equation, it follows that there cannot be a linear element of the form

$$d\sigma^2 = \frac{1}{(U_1 + V_1)}(du_1^2 + dv_1^2),$$

with neither U_1 nor V_1 constant. We shall consider now this exceptional case and assume that V_1 is constant. In this case the curves on the sphere are great circles with a common diameter and their orthogonal trajectories.

* Bianchi, Lezioni, p. 67.

With respect to the corresponding surface, we have only to note that by retracing the steps which led to the linear element of the sphere in the above form, we find that when V_1 is a constant, V also is a constant. Then both ρ_1 and ρ_2 are functions of u alone, and consequently the surfaces are surfaces of revolution. From the well-known theorem of Bonnet with regard to the relation between surfaces of constant mean curvature and those with constant total curvature, one has that the surfaces of revolution of constant mean curvature are parallels of the surfaces of revolution with constant positive total curvature. Since the latter are known, the same is true of the former. Combining these results with those previously established with respect to the sphere and minimal surfaces, we have the theorem :

The only surfaces of constant mean curvature whose lines of curvature are isothermal-conjugate are the sphere, minimal surfaces and the surfaces of revolution whose mean curvature is constant.

We will deduce now the differential relation which exists between the principal radii of curvature of a surface whose lines of curvature are isothermal-conjugate. To this end we consider the Codazzi equations and the equation of Gauss, which, when the surface is referred to its lines of curvature reduce to*

$$\frac{\partial}{\partial v}\left(\frac{D}{\sqrt{E}}\right) - \frac{D''}{G}\,\frac{\partial\sqrt{E}}{\partial v} = 0, \quad \frac{\partial}{\partial u}\left(\frac{D''}{\sqrt{G}}\right) - \frac{D}{E}\,\frac{\partial\sqrt{G}}{\partial u} = 0, \tag{15}$$

$$\frac{DD''}{\sqrt{EG}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\,\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\,\frac{\partial\sqrt{E}}{\partial v}\right) = 0. \tag{16}$$

Now
$$E = \rho_1 D, \quad G = \rho_2 D'',$$

hence, equations (15) may be written

$$\left.\begin{aligned} \frac{\partial \log\sqrt{D}}{\partial v} &= \frac{\rho_2+\rho_1}{\rho_2-\rho_1}\,\frac{\partial \log\sqrt{\rho_1}}{\partial v}, \\ \frac{\partial \log\sqrt{D''}}{\partial u} &= -\frac{\rho_2+\rho_1}{\rho_2-\rho_1}\,\frac{\partial \log\sqrt{\rho_2}}{\partial u}. \end{aligned}\right\} \tag{17}$$

Expressing the condition that the lines of curvature are isothermal-conjugate, that is,

$$\frac{D}{D''} = \frac{U}{V},$$

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* Bianchi, p. 93.

we have the following condition which ρ_1 and ρ_2 must satisfy:

$$\frac{\partial}{\partial u}\left(\frac{\rho_1+\rho_2}{\rho_1-\rho_2}\frac{\partial}{\partial v}\log\sqrt{\rho_1}\right)+\frac{\partial}{\partial v}\left(\frac{\rho_1+\rho_2}{\rho_1-\rho_2}\frac{\partial}{\partial u}\log\sqrt{\rho_2}\right)=0,$$

or, on developing,

$$\frac{1}{\rho_1}\frac{\partial^2\rho_1}{\partial u\,\partial v}+\frac{1}{\rho_2}\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{\rho_1^2+2\rho_1\rho_2-\rho_2^2}{\rho_1^2(\rho_1^2-\rho_2^2)}\frac{\partial\rho_1}{\partial u}\frac{\partial\rho_1}{\partial v}$$
$$+\frac{\rho_2^2+2\rho_1\rho_2-\rho_1^2}{\rho_2^2(\rho_1^2-\rho_2^2)}\frac{\partial\rho_2}{\partial u}\frac{\partial\rho_2}{\partial v}=0, \tag{18}$$

which is the relation sought. If we put

$$\phi=\frac{1}{\rho_1\rho_2},\quad \psi=\frac{\rho_2-\rho_1}{\rho_1\rho_2(\rho_1+\rho_2)}, \tag{19}$$

then the above equation takes the simple form

$$2\psi\frac{\partial^2\phi}{\partial u\,\partial v}-\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial v}-\frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial u}=0, \tag{20}$$

and the Codazzi equations become

$$\left.\begin{aligned}\psi\frac{\partial}{\partial v}\log D+\frac{\partial\phi}{\partial v}+\frac{\partial\psi}{\partial v}=0,\\ \psi\frac{\partial}{\partial u}\log D-\frac{\partial\phi}{\partial u}+\frac{\partial\psi}{\partial u}=0.\end{aligned}\right\} \tag{21}$$

From these equations we find

$$D=\frac{1}{\psi}e^{\int\frac{1}{\psi}\left(\frac{\partial\phi}{\partial u}du-\frac{\partial\phi}{\partial v}dv\right)}. \tag{22}$$

Making use of equations (15) and putting

$$D=\varepsilon D''.$$

where ε is $+1$ or -1, according as S is a surface of positive or negative curvature, we can put equation (16) in the form

$$\frac{D}{\sqrt{\varepsilon\rho_1\rho_2}}+\frac{\partial}{\partial u}\left[\sqrt{\varepsilon\frac{\rho_1}{\rho_2}}\,\frac{\rho_2}{\rho_2-\rho_1}\,\frac{\partial}{\partial u}\log\frac{1}{\rho_2}\right]$$
$$+\frac{\partial}{\partial v}\left[\sqrt{\varepsilon\frac{\rho_2}{\rho_1}}\,\frac{\rho_1}{\rho_1-\rho_2}\,\frac{\partial}{\partial v}\log\frac{1}{\rho_1}\right]=0. \tag{23}$$

III.

Consider the quadric surface whose equation is

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1. \tag{1}$$

The cartesian coordinates have the following expressions in terms of parameters referring to the lines of curvature :*

$$\left.\begin{aligned} x &= \sqrt{\frac{a(a-u)(a-v)}{(a-b)(a-c)}}, \\ y &= \sqrt{\frac{b(b-u)(b-v)}{(b-a)(b-c)}}, \\ z &= \sqrt{\frac{c(c-u)(c-v)}{(c-a)(c-b)}}. \end{aligned}\right\} \tag{2}$$

Calculating the direction cosines of the tangent plane to the surface at the point (x, y, z) and the coefficients of the first and second fundamental forms, we get

$$X = \sqrt{\frac{bc}{uv}}\sqrt{\frac{(a+u)(a+v)}{(a-b)(a-c)}}, \quad Y = \sqrt{\frac{ca}{uv}}\sqrt{\frac{(b+u)(b+v)}{(b-c)(b-a)}},$$
$$Z = \sqrt{\frac{ab}{uv}}\sqrt{\frac{(c+u)(c+v)}{(c-a)(c-b)}}. \tag{3}$$

$$E = \frac{u(u-v)}{4(a+u)(b+u)(c+u)}, \quad G = \frac{v(u-v)}{4(a+v)(b+v)(c+v)}, \tag{4}$$

$$D = -\tfrac{1}{4}\sqrt{\frac{abc}{uv}} \; \frac{u-v}{(a+u)(b+u)(c+u)},$$
$$D'' = -\tfrac{1}{4}\sqrt{\frac{abc}{uv}} \; \frac{u-v}{(a+v)(b+v)(c+v)}. \tag{5}$$

From (4) we remark that the surfaces of the second degree whose equation is of the form (1) are isothermic and the expressions (5) show that *the lines of curvature of quadric surfaces are isothermal-conjugate.*

The surfaces of the second degree whose center is at an infinite distance can be looked upon as limiting cases of the preceding, and from this point of view

* Darboux, Leçons, t. I, p. 156.

we see that their lines of curvature are isothermal-conjugate. But we can prove it directly from the expressions for the cartesian coordinates of such surfaces. Thus, if the equation of the surface is

$$ax^2 + by^2 + 2cz + d = 0,$$

we have for the coordinates,*

$$\left.\begin{aligned} x &= c\sqrt{\frac{(a-b)}{b\,(ub+a)(vb+a)}}, \\ y &= c\sqrt{\frac{b-a}{a\,(ub+a)(vb+a)}}, \\ z &= \frac{c\,(a-b)}{2a}\,\frac{b^2\,uv - a^2}{(ub+a)(vb+a)} - \frac{d}{2c}. \end{aligned}\right\} \qquad (6)$$

Proceeding as in the former case, we find

$$X,\ Y,\ Z = \frac{a^2\sqrt{\dfrac{a-b}{a}},\quad ib^2\sqrt{\dfrac{a-b}{b}}\sqrt{uv},\quad \sqrt{ab\,(ub+a)(vb+a)}}{\sqrt{(b^2u+a^2)(b^2v+a^2)}}, \qquad (7)$$

$$E = \tfrac{1}{4}\,\frac{c^2\,(a-b)(u-v)\left(b^2+\dfrac{a^2}{u}\right)}{(ub+a)^4\,(vb+a)}, \quad G = -\tfrac{1}{4}\,\frac{c^2(a-b)(u-v)\left(b^2+\dfrac{a^2}{v}\right)}{(ub+a)(vb+a)^3}, \qquad (8)$$

$$D = \tfrac{1}{4}\,\frac{(ab)^{\frac{3}{2}}\,c\,(a-b)(u-v)}{\sqrt{(b^2u+a^2)(b^2v+a^2)(bu+a)(bv+a)}}\cdot\frac{1}{u\,(ub+a)^2}, \qquad (9)$$

$$D'' = \tfrac{1}{4}\,\frac{(ab)^{\frac{3}{2}}\,c\,(a-b)(u-v)}{\sqrt{(b^2u+a^2)(b^2v+a^2)(bu+a)(bv+a)}}\cdot\frac{1}{v\,(vb+a)^2}.$$

From these forms we see that *the lines of curvature of quadrics with their centers at an infinite distance are isothermal and isothermal-conjugate.*

Surfaces of revolution can be defined by the equations

$$x = r\cos v, \quad y = r\sin v, \quad z = \phi(r).$$

Then the square of the linear element takes the form

$$ds^2 = [1 + \phi'(r)^2]\,dr^2 + r^2\,dv^2,$$

and it is readily found that the second fundamental form is

$$f = \frac{\phi''(r)\,dr^2 - r\phi'(r)\,dv^2}{\sqrt{1+\phi'(r)^2}}.$$

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* Willgrod, ib., p. 27.

Hence, *surfaces of revolution have isothermal-conjugate lines of curvature*, and, moreover, the ratio of D and D'' is a function of r alone.

IV.

In this section we wish to determine those surfaces which, together with their parallels, have isothermal-conjugate lines of curvature.

In solving this problem, we make use of the equation

$$\frac{1}{\rho_1}\frac{\partial^2\rho_1}{\partial u\,\partial v}+\frac{1}{\rho_2}\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{\rho_1^2+2\rho_1\rho_2-\rho_2^2}{\rho_1^2(\rho_1^2-\rho_2^2)}\frac{\partial\rho_1}{\partial u}\frac{\partial\rho_1}{\partial v}+\frac{\rho_2^2+2\rho_1\rho_2-\rho_1^2}{\rho_2^2(\rho_1^2-\rho_2^2)}\frac{\partial\rho_2}{\partial u}\frac{\partial\rho_2}{\partial v}=0, \qquad (1)$$

which, as we have seen, is the condition which the principal radii of curvature of a surface S must satisfy in order that its lines of curvature be isothermal-conjugate. We have then that this equation must also be satisfied by ρ_1+a and ρ_2+a ($a=\text{const.}$), for all values of a, in order that the lines of curvature on the parallels of S may form an isothermal system. Since this condition must be satisfied identically for values of a, it is only necessary to substitute ρ_1+a, ρ_2+a for ρ_1 and ρ_2 respectively and equate to zero the coefficients of the different powers of a. This gives

$$\frac{\partial^2\rho_1}{\partial u\,\partial v}+\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{1}{\rho_1-\rho_2}\left(\frac{\partial\rho_1}{\partial u}\frac{\partial\rho_1}{\partial v}-\frac{\partial\rho_2}{\partial u}\frac{\partial\rho_2}{\partial v}\right)=0,$$

$$(3\rho_1+5\rho_2)\frac{\partial^2\rho_1}{\partial u\,\partial v}+(5\rho_1+3\rho_2)\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{4\rho_1+\rho_2}{\rho_1-\rho_2}\left(\frac{\partial\rho_1}{\partial u}\frac{\partial\rho_1}{\partial v}-\frac{\partial\rho_2}{\partial u}\frac{\partial\rho_2}{\partial v}\right)=0,$$

$$(\rho_1^2+4\rho_2^2+7\rho_1\rho_2)\frac{\partial^2\rho_1}{\partial u\,\partial v}+(\rho_2^2+4\rho_1^2+7\rho_1\rho_2)\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{\rho_1^2+10\rho_1\rho_2+\rho_2^2}{\rho_1-\rho_2}\left(\frac{\partial\rho_1}{\partial u}\frac{\partial\rho_1}{\partial v}-\frac{\partial\rho_2}{\partial u}\frac{\partial\rho_2}{\partial v}\right)=0.$$

$$\rho_2(2\rho_1^2+5\rho_1\rho_2+\rho_2^2)\frac{\partial^2\rho_1}{\partial u\,\partial v}+\rho_1(\rho_1^2+5\rho_1\rho_2+2\rho_2^2)\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{2}{\rho_1-\rho_2}\left[\rho_2(4\rho_1\rho_2+\rho_1^2-\rho_2^2)\frac{\partial\rho_1}{\partial u}\frac{\partial\rho_1}{\partial v}-\rho_1(4\rho_1\rho_2+\rho_2^2-\rho_1^2)\frac{\partial\rho_2}{\partial u}\frac{\partial\rho_2}{\partial v}\right]=0,$$

$$\frac{1}{\rho_1}\frac{\partial^2\rho_1}{\partial u\,\partial v}+\frac{1}{\rho_2}\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{1}{\rho_1^2\rho_2^2(\rho_1^2-\rho_2^2)}\left[\rho_2^2(\rho_1^2+2\rho_1\rho_2-\rho_2^2)\frac{\partial\rho_1}{\partial u}\frac{\partial\rho_1}{\partial v}-\rho_1^2(\rho_2^2+2\rho_1\rho_2-\rho_1^2)\frac{\partial\rho_2}{\partial u}\frac{\partial\rho_2}{\partial v}\right]=0.$$

Combining the first two of the above equations, we find that either

$$\frac{\partial^2\rho_1}{\partial u\,\partial v}-\frac{\partial^2\rho_2}{\partial u\,\partial v}=0,$$

or

$$\rho_1-\rho_2=0.$$

The latter simply gives the case of the sphere, which is an evident solution of the problem. Taking the former and proceeding step by step with the above equations, we are brought to the following equations of condition, which must be satisfied by surfaces furnishing a solution of the problem:

$$\frac{\partial^2\rho_1}{\partial u\,\partial v}=0,\quad \frac{\partial^2\rho_2}{\partial u\,\partial v}=0,\quad \frac{\partial\rho_1}{\partial u}\,\frac{\partial\rho_1}{\partial v}=0,\quad \frac{\partial\rho_2}{\partial u}\,\frac{\partial\rho_2}{\partial v}=0. \tag{2}$$

There are four possible solutions of this system of equations; they are

$$\left.\begin{array}{lll} 1^\circ. & \rho_1=f(u), & \rho_2=\phi(v), \\ 2^\circ. & \rho_1=f(v), & \rho_2=\phi(u), \\ 3^\circ. & \rho_1=f(u), & \rho_2=\phi(u), \\ 4^\circ. & \rho_1=f(v), & \rho_2=\phi(v). \end{array}\right\} \tag{3}$$

Since the last two cases are practically identical, we have only the first three to consider. We have found that for surfaces with isothermal-conjugate lines of curvature the following relation exists:

$$\frac{D}{\sqrt{\varepsilon\rho_1\rho_2}}+\frac{\partial}{\partial u}\left[\sqrt{\varepsilon\frac{\rho_1}{\rho_2}}\,\frac{\rho_2}{\rho_2-\rho_1}\,\frac{\partial}{\partial u}\log\frac{1}{\rho_2}\right] +\frac{\partial}{\partial v}\left[\sqrt{\frac{\varepsilon\rho_2}{\rho_1}}\,\frac{\rho_1}{\rho_1-\rho_2}\,\frac{\partial}{\partial v}\log\frac{1}{\rho_1}\right]=0. \tag{4}$$

$$1^\circ.\quad \rho_1=f(u),\qquad \rho_2=\phi(v).$$

From (4) it is seen that either

$$\rho_1=\rho_2=\text{const.},$$

or

$$D=0.$$

Hence the sphere and the plane are solutions of the problem.

$$2^\circ.\quad \rho_1=f(v),\qquad \rho_2=\phi(u).$$

Since, by definition, ρ_1 is the radius of curvature of the normal section tangent to the curves $v=\text{const.}$, we see that for this class of surfaces ρ_1 is constant along such a curve. Hence, the normals to the surface along such a curve meet in a

point, and, consequently, form a cone, and the curve lies on the sphere whose center is at the vertex of the cone and of radius ρ_1. It is evident, therefore, that the surface is the envelope of a family of spheres of radius $\rho_1 = f(v)$. In a similar way it can be shown that the surface is the envelope of a family of spheres of radius $\phi(u)$. Hence, the surfaces of the case under discussion are *cyclides of Dupin.* The coefficients of the second fundamental form of any surface of this class can be found at once from (4),

$$3^\circ. \quad \rho_1 = f(u), \qquad \rho_2 = \phi(u).$$

The surfaces corresponding to these expressions are evidently surfaces of revolution; from (4) we find

$$D = \varepsilon D'' = F(u),$$

where ε is $+1$ or -1, according as S has positive or negative curvature.

Hence, *the only surfaces which, together with their parallels, have isothermal-conjugate lines of curvature, are the plane, sphere, cyclides of Dupin and surfaces of revolution.*

Combining (3) and

$$\rho_1 = \frac{E}{D}, \quad \rho_2 = \frac{G}{D''},$$

we see that since $D'' = \varepsilon D$,

$$\frac{E}{G} = \frac{U}{V},$$

hence all of these surfaces are isothermic. Conversely, from the results found by Willgrod in solving the similar problem for isothermic surfaces, we remark that all surfaces which, together with their parallels are isothermic, have isothermal-conjugate lines of curvature and their parallels have the same property.

V.

We will determine the surfaces whose principal radii satisfy a relation and whose lines of curvature form an isothermal-conjugate system.

In solving this problem, we make use of the following equation, which we have found must be satisfied in order that the lines of curvature of the corresponding surface be isothermal-conjugate:

$$2\psi \frac{\partial^2 \phi}{\partial u \partial v} - \frac{\partial \phi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \phi}{\partial v} \frac{\partial \psi}{\partial u} = 0, \tag{1}$$

where
$$\phi = \frac{1}{\rho_1 \rho_2}, \quad \psi = \frac{\rho_2 - \rho_1}{\rho_1 \rho_2 (\rho_1 + \rho_2)}. \tag{2}$$

If ρ_1 is a function of ρ_2, then ψ will be a function of ϕ except when the relation between the radii is

$$\frac{1}{\rho_1 \rho_2} = \text{const.} \quad \text{or} \quad \frac{\rho_2 - \rho_1}{\rho_1 \rho_2 (\rho_1 + \rho_2)} = \text{const.}$$

From (1) it is seen that the former of these exceptional cases gives a solution of our problem. In order that the second may give a solution, we must have

$$\phi = \frac{1}{\rho_1 \rho_2} = U + V,$$

where U is a function of u alone and V is a function of v alone. From (II, 22) it follows that

$$D = \frac{1}{C} e^{\frac{U - V}{C}}.$$

The functions U and V must be of such a character as to satisfy the transformed Gauss equation

$$\frac{D}{\sqrt{\epsilon \rho_1 \rho_2}} + \frac{\partial}{\partial u}\left[\sqrt{\frac{\epsilon \rho_1}{\rho_2}} \frac{\rho_2}{\rho_2 - \rho_1} \frac{\partial}{\partial u} \log \frac{1}{\rho_2}\right]$$
$$+ \frac{\partial}{\partial v}\left[\sqrt{\frac{\epsilon \rho_2}{\rho_1}} \frac{\rho_1}{\rho_1 - \rho_2} \frac{\partial}{\partial v} \log \frac{1}{\rho_1}\right] = 0. \tag{3}$$

Proceed now to the general case where ψ is a function of ϕ; then

$$\frac{\partial \psi}{\partial u} = \frac{\partial \psi}{\partial \phi} \frac{\partial \phi}{\partial u}, \quad \frac{\partial \psi}{\partial v} = \frac{\partial \psi}{\partial \phi} \frac{\partial \phi}{\partial v}.$$

Hence, equation (1) reduces to

$$\psi \frac{\partial^2 \phi}{\partial u \partial v} - \frac{\partial \psi}{\partial \phi} \frac{\partial \phi}{\partial u} \frac{\partial \phi}{\partial v} = 0,$$

or
$$\frac{\partial}{\partial v}\left(\frac{\frac{\partial \phi}{\partial u}}{\psi}\right) = 0, \quad \frac{\partial}{\partial u}\left(\frac{\frac{\partial \phi}{\partial v}}{\psi}\right) = 0,$$

If, now, we write
$$W = U + V,$$

where U is a function of u alone and V is a function of v alone, we have from the above equation

$$\frac{\partial \phi}{\partial u} = \psi \frac{\partial W}{\partial u}, \quad \frac{\partial \phi}{\partial v} = \psi \frac{\partial W}{\partial v}, \tag{4}$$

whence

$$W = \int \frac{d\phi}{\psi}.$$

Hence ϕ and ψ are functions of W; from (4) we see that these functions are such that ψ is the derivative of ϕ with respect to W. We can write therefore

$$\phi = \frac{1}{\rho_1 \rho_2} = f(W), \quad \psi = \frac{\rho_2 - \rho_1}{\rho_1 \rho_2 (\rho_1 + \rho_2)} = f'(W). \tag{5}$$

From (II, 22) we find

$$D = \frac{e^{U-V}}{f'(W)}. \tag{6}$$

The functions f, U, V must be such as to verify equation (3), which, on replacing ρ_1, ρ_2, D by their expressions from (5) and (6), becomes

$$\frac{e^{U-V}.\sqrt{f}}{f'} + \frac{\partial}{\partial u}\left[\frac{U'}{2}\sqrt{\frac{f-f'}{f+f'}}\left(\frac{f'}{f} + \frac{f'-f''}{f-f'} - \frac{f'+f''}{f+f'} + \frac{2(f'-f'')}{f'}\right)\right]$$
$$+ \frac{\partial}{\partial v}\left[\frac{V'}{2}\sqrt{\frac{f+f'}{f-f'}}\left(\frac{f'}{f} + \frac{f'+f''}{f+f'} - \frac{f'-f''}{f-f'} - \frac{2(f'+f'')}{f'}\right)\right] = 0. \tag{7}$$

It is evident that surfaces of revolution correspond to the case when ϕ and ψ are functions of only one of the parameters.

From the theorem of Weingarten,* we have that for surfaces whose principal radii satisfy a relation, the parameters (u, v) of the lines of curvature *can be so chosen* that the square of the linear element of the spherical representation of the surface takes the form

$$ds'^2 = \frac{du^2}{\alpha^2} + \frac{dv^2}{\theta'^2(\alpha)}, \tag{8}$$

where α is a function of u, v, and the character of the function θ depends upon the relation between the radii; this function θ being such that

$$\rho_1 = \theta(\alpha), \quad \rho_2 = \theta(\alpha) - \alpha\theta'(\alpha). \tag{9}$$

From this it is seen that by changing α we get different members of a family of surfaces whose radii of curvature satisfy the same relation. We wish to find

* "Ueber die Oberflächen, für welche einer der beiden Hauptkrümmungshalbmesser eineFunction des andern ist." Crelle, Vol. 62, pp. 160-173.

whether there are any such families, whose lines of curvature are isothermal-conjugate and in such a way that for the parameters, as above chosen,

$$D = \varepsilon D''.$$

Consider first the case $D = D''$; it is readily shown that in consequence of this we have

$$\mathfrak{E}\rho_1 = \mathfrak{G}\rho_2.$$

Replacing the quantities in this equation by their expressions as given by (8) and (9), we get

$$\frac{\theta(\alpha)}{\alpha^2} = \frac{\theta(\alpha) - \alpha\theta'(\alpha)}{\theta''(\alpha)}.$$

Hence, the solution of our problem reduces to the integration of this equation. It can readily be brought to Clairant's form, from which we get the general solution

$$\theta^2(\alpha) = c\alpha^2 + c^2;$$

this gives

$$\rho_1 = c\sqrt{1 + \frac{\alpha^2}{c}}, \quad \rho_2 = \frac{c}{\sqrt{1 + \frac{\alpha^2}{c}}},$$

hence, by elimination, we find that the surfaces for which

$$\rho_1\rho_2 = c^2,$$

furnish the solution.

Proceeding in a similar manner for surfaces of negative curvature, we find

$$\rho_1 = c\sqrt{\frac{\alpha^2}{c} - 1}, \qquad \rho_2 = -\frac{c}{\sqrt{\frac{\alpha^2}{c} - 1}},$$

that is,

$$\rho_1\rho_2 = -c^2.$$

Hence, *the surfaces of constant curvature furnish the general solution of our problem.*

VI.

Consider a surface whose lines of curvature are plane in both systems, and let the planes of these lines have for equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \end{aligned}\right\} \tag{1}$$

where a_1, b_1, c_1, d_1 are functions of u alone and a_2, b_2, c_2, d_2 are functions of v alone—u, v being parameters of the lines of curvature.

Differentiating these equations with respect to u, we have

$$\left.\begin{aligned} a_1\frac{\partial x}{\partial u}+b_1\frac{\partial y}{\partial u}+c_1\frac{\partial z}{\partial u}&=-(xa_1'+yb_1'+zc_1'-d_1'),\\ a_2\frac{\partial x}{\partial u}+b_2\frac{\partial y}{\partial u}+c_2\frac{\partial z}{\partial u}&=0,\end{aligned}\right\} \tag{2}$$

where accents denote differentiation with respect to u; in like manner,

$$\left.\begin{aligned} a_2\frac{\partial x}{\partial v}+b_2\frac{\partial y}{\partial v}+c_2\frac{\partial z}{\partial v}&=-(xa_2'+yb_2'+zc_2'-d_2'),\\ a_1\frac{\partial x}{\partial v}+b_1\frac{\partial y}{\partial v}+c_1\frac{\partial z}{\partial v}&=0,\end{aligned}\right\} \tag{3}$$

where the accents denote differentiation with respect to v. Combining equations (2) with

$$X\frac{\partial x}{\partial u}+Y\frac{\partial y}{\partial u}+Z\frac{\partial z}{\partial u}=0, \tag{4}$$

and solving for $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$, we find

$$\left.\begin{aligned} \frac{\partial x}{\partial u}&=-\frac{(xa_1'+yb_1'+zc_1'-d_1')(b_2Z-c_2Y)}{\Delta},\\ \frac{\partial y}{\partial u}&=-\frac{(xa_1'+yb_1'+zc_1'-d_1')(c_2X-a_2Z)}{\Delta},\\ \frac{\partial z}{\partial u}&=-\frac{(xa_1'+yb_1'+zc_1'-d_1')(a_2Y-b_2X)}{\Delta},\end{aligned}\right\} \tag{5}$$

where Δ denotes the determinant of the system of equations. From these expressions we find

$$D=-\Sigma\frac{\partial x}{\partial u}\frac{\partial X}{\partial u}=\frac{(xa_1'+yb_1'+zc_1'-d_1')}{\Delta}\Big[a_2\Big(Y\frac{\partial Z}{\partial u}-Z\frac{\partial Y}{\partial u}\Big)$$
$$+b_2\Big(Z\frac{\partial X}{\partial u}-X\frac{\partial Z}{\partial u}\Big)+c_2\Big(X\frac{\partial Y}{\partial u}-Y\frac{\partial X}{\partial u}\Big)\Big]. \tag{6}$$

By treating equations (3) and

$$X\frac{\partial x}{\partial v}+Y\frac{\partial y}{\partial v}+Z\frac{\partial z}{\partial v}=0$$

in a similar manner, we find

$$D'' = \frac{(xa_2' + yb_2' + zc_2' - d_2')}{\Delta}\Big[a_1\Big(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\Big)$$
$$+ b_1\Big(Z\frac{\partial X}{\partial v} - X\frac{\partial Z}{\partial v}\Big) + c_1\Big(X\frac{\partial Y}{\partial v} - Y\frac{\partial X}{\partial v}\Big)\Big]. \quad (7)$$

Hence the necessary and sufficient condition that the lines of curvature on such a surface form an isothermal-conjugate system is that the functions $a_1, a_2, \ldots, d_2$ satisfy the relation

$$\frac{(xa_1' + yb_1' + zc_1' - d_1')\Big[a_2\Big(Y\frac{\partial Z}{\partial u} - Z\frac{\partial Y}{\partial u}\Big) + b_2\Big(Z\frac{\partial X}{\partial u} - X\frac{\partial Z}{\partial u}\Big) + c_2\Big(X\frac{\partial Y}{\partial u} - Y\frac{\partial X}{\partial u}\Big)\Big]}{(xa_2' + yb_2' + zc_2' - d_2')\Big[a_1\Big(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\Big) + b_1\Big(Z\frac{\partial X}{\partial v} - X\frac{\partial Z}{\partial v}\Big) + c_1\Big(X\frac{\partial Y}{\partial v} - Y\frac{\partial X}{\partial v}\Big)\Big]} = \frac{U}{V}. \quad (8)$$

where U is a function of u alone and V is a function of v alone.

We will recall that all surfaces with plane lines of curvature in both systems can be divided into three families:*

1°. *Moulure surfaces* and *surfaces of revolution.* The surfaces of this family may be defined as the envelopes of the plane

$$uz - x\cos v - y\sin v = f(u) + \phi(v), \quad (9)$$

for different forms of the functions f and ϕ. It is manifest that the surfaces of revolution correspond to $\phi(v) = 0$;

2°. Surfaces defined as the envelope of the plane

$$ux - vy + (\lambda\sqrt{1 - u^2} - \sqrt{\lambda^2 - 1}\,.\,\sqrt{1 + v^2})z = f(u) - \phi(v); \quad (10)$$

3°. Surfaces defined as the envelope of the plane

$$2ux + 2vy + (1 - u^2 - v^2)z = f(u) + \phi(v). \quad (11)$$

Geometrically, the three families are distinguished by their spherical representation.† Those of the first family have for spherical representation a system of great circles pivoting about a diameter and their orthogonal parallels. The

* Darboux, Leçons, t. I, p. 130. † Caronnet, Thesis, p. 10,

lines of curvature of the surfaces of the second family have for their spherical representation a double system of small circles whose respective planes meet in two reciprocal straight lines; when these lines are tangent to the sphere, the spherical representation is that of the third family. For a method of point and tangential generation of all surfaces with plane lines of curvature in both systems the reader is referred to Darboux.*

We will now discuss each of these families.

Surfaces of the First Family.

As previously remarked, these surfaces may be defined as the envelope of the plane

$$uz - x \cos v - y \sin v = f(u) + \phi(v). \tag{9}$$

On taking the derivatives of this equation with respect to u and v respectively, we get the equations of the planes of the lines $u =$ const. and $v =$ const.; they are

$$\left. \begin{aligned} z &= f'(u), \\ x \sin v - y \cos v &= \phi'(v). \end{aligned} \right\} \tag{12}$$

Comparing these equations with equations (1), we remark that for this case

$$\left. \begin{aligned} &a_1 = b_1 = 0, \quad c_1 = 1, \quad d_1 = f'(u), \\ &a_2 = \sin v, \quad b_2 = -\cos v, \quad c_2 = 0, \quad d_2 = \phi'(v); \end{aligned} \right\} \tag{13}$$

and from (9),

$$X = \frac{-\cos v}{\sqrt{1+u^2}}, \quad Y = -\frac{\sin v}{\sqrt{1+u^2}}, \quad Z = \frac{u}{\sqrt{1+u^2}}.$$

When these expressions are substituted in the conditional equation (8), it is found that $f(u)$ and $\phi(v)$ must be such that

$$\frac{f''(u)}{x \cos v + y \sin v - \phi''(v)} = \frac{U}{V},$$

which, in consequence of equations (9) and (12), can be written

$$\frac{1}{uf'(u) - f(u) - \phi(v) - \phi''(v)} = \frac{U_2}{V}, \tag{14}$$

where now $U_2 = \dfrac{U}{f''(u)}$. In order that this relation may be satisfied either

* Leçons, t. I, p. 132.

1°. $uf'(u) - f(u) = C_1$ and $\phi(v)$ has any form whatever ; or,

2°. $\phi''(v) + \phi(v) = C_2$ and $f(u)$ has any form, where C_1 and C_2 are constants.

In the former case, we would have

$$f(u) = C_3 u - C_1. \tag{15}$$

However, in this case,

$$z = f'(u) = C_3,$$

that is, the surface would be a plane curve, and hence this case does not hold.

In the second case,

$$\phi(v) = (c_1 + c_2)\cos v + i(c_1 - c_2)\sin v + C_2, \tag{16}$$

where c_1 and c_2 are arbitrary constants.

We have then for surfaces of the first family whose lines of curvature form an isothermal-conjugate system, *surfaces of revolution* and those *moulure surfaces* for which $\phi(v)$ has the form (16) and $f(u)$ has any form with the exception of (15).

Surfaces of the Second Family.

These surfaces being the envelopes of the plane

$$ux - vy + (\lambda\sqrt{1-u^2} - \sqrt{\lambda^2 - 1}\sqrt{1+v^2})z = f(u) - \phi(v), \tag{10}$$

the planes of their lines of curvature are

$$\left.\begin{aligned} x - \frac{\lambda u}{\sqrt{1-u^2}} z &= f'(u), \\ y + \frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}} z &= \phi'(v). \end{aligned}\right\} \tag{17}$$

Comparing these with equations (1), we have for this case

$$\left.\begin{aligned} a_1 = 1,\ b_1 = 0,\ c_1 &= -\frac{\lambda u}{\sqrt{1-u^2}},\quad d_1 = f'(u), \\ a_2 = 0,\ b_2 = 1,\ c_2 &= \frac{\sqrt{\lambda^2-1}\,.\,v}{\sqrt{1+v^2}},\quad d_2 = \phi'(v), \end{aligned}\right\} \tag{18}$$

and from (10),

$$X,\ Y,\ Z = \frac{u,\ -v,\ \lambda\sqrt{1-u^2} - \sqrt{\lambda^2-1}\,.\sqrt{1+v^2}}{(u^2(1-\lambda^2) + \lambda^2 v^2 + 2\lambda^2 - 1 - 2\lambda\sqrt{\lambda^2-1}\sqrt{1-u^2}\,.\sqrt{1-v^2})^{\frac{1}{2}}}. \tag{18'}$$

When these values are substituted in equation (8), the latter becomes

$$\frac{f''(u) + \lambda\left(\frac{u}{\sqrt{1-u^2}}\right)'z}{\phi''(v) - \sqrt{\lambda^2-1}\left(\frac{v}{\sqrt{1-v^2}}\right)'z} = \frac{U}{V}. \tag{19}$$

Replacing z by its expression, as determined from (10) and (17), viz.:

$$z = \frac{[f(u) - uf'(u)] - [\phi(v) - v\phi'(v)]}{\frac{\lambda}{\sqrt{1-u^2}} - \frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}}},$$

we can put the equation in the form

$$\frac{\frac{f''(u)}{\lambda\left(\frac{u}{\sqrt{1-u^2}}\right)}\left(\frac{\lambda}{\sqrt{1-u^2}} - \frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}}\right) + f(u) - uf'(u) - \phi(v) + v\phi'(v)}{\frac{\phi''(v)}{\sqrt{\lambda^2-1}\,.\left(\frac{v}{\sqrt{1+v^2}}\right)}\left(\frac{\lambda}{\sqrt{1-u^2}} - \frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}}\right) - f(u) + uf'(u) + \phi(v) - v\phi'(v)} = \frac{U_1}{V_1}, \tag{20}$$

where U_1 and V_1 are new functions of u and v respectively.

But this equation, which determines the character of f and ϕ so that the lines of curvature on the corresponding surface forms an *isothermal-conjugate* system, is exactly the same equation which Caronnet* has found in seeking to determine what surfaces of the second family are *isothermic*. We have then only to recall his results in order to have an answer to our problem, at the same time referring the reader to the elegant method used by Caronnet for the solution of the above equation. They are as follows:

$$\left.\begin{aligned} f(u) &= l\sqrt{1-u^2} + mu + n, \\ \phi(v) &= l'\sqrt{1+v^2} + m'v + n', \end{aligned}\right\} \tag{I}$$

where l m, n, l', m', n' are any constants whatever. The surface corresponding to these forms of f and ϕ is the *cyclide of Dupin* of the *fourth* order.

$$\left.\begin{aligned} f(u) &= \lambda^2\left(1 - \frac{u}{2}\log\frac{1+u}{1-u}\right), \\ \phi(v) &= (\lambda^2-1)(1 + v\tan^{-1}v). \end{aligned}\right\} \tag{II}$$

* These—"Recherches sur les surfaces isothermiques et les surfaces dont les rayons de courbure sont fonctions l'un de l'autre, p. 13.

$$\left.\begin{aligned} f(u) &= -\frac{2t}{1+t^2}\left[\frac{t}{\lambda+c} - \frac{1}{(\lambda-c)t} - \frac{4\lambda^2}{(\lambda^2-c^2)^{\frac{3}{2}}}\tan^{-1}\sqrt{\frac{\lambda+c}{\lambda-c}}\,t\right], \\ \phi(v) &= -\frac{2\tau}{1+\tau^2}\left[\frac{\tau}{\sqrt{\lambda^2-1}+c} - \frac{1}{\sqrt{(\lambda^2-1-c)}\,\tau}\right. \\ &\qquad \left. - \frac{4(\lambda^2-1)}{(\lambda^2-1-c^2)^{\frac{3}{2}}}\tan^{-1}\sqrt{\frac{\sqrt{\lambda^2-1}+c}{\sqrt{\lambda^2-1}-c}}\,\tau\right], \end{aligned}\right\} \text{(III)}$$

where, in order to simplify the expressions, we have written

$$u = \frac{2t}{1+t^2}, \quad v = \frac{2i\tau}{1+\tau^2},$$

and c is an arbitrary constant. When, in particular $c = 0$, the functions define the *minimal surface of Bonnet*. In determining the above forms for f and ϕ, it was assumed that the product

$$(\lambda^2 - c^2)(\lambda^2 - 1 - c^2)$$

is different from zero. In discussing this special case, there is need only of considering that where $\lambda^2 - c^2 = 0$ on account of symmetry. This gives

$$\left.\begin{aligned} f(u) &= -\frac{3t^4 + 6t^2 - 1}{3\lambda t^2(1+t^2)}, \\ \phi(v) &= -\frac{2\tau}{1+\tau^2}\left[\frac{\tau}{\sqrt{\lambda^2-1}+\lambda} - \frac{1}{(\sqrt{\lambda^2-1}-\lambda)\tau}\right. \\ &\qquad \left. - \frac{4(\lambda^2-1)}{i}\tan^{-1}\sqrt{\frac{\sqrt{\lambda^2-1}+\lambda}{\sqrt{\lambda^2-1}-\lambda}}\,\tau\right]. \end{aligned}\right\} \text{(III')}$$

Finally

$$\left.\begin{aligned} f(u) &= \frac{\lambda^2\sqrt{\alpha^2-\lambda^2}}{\alpha}\int\frac{\sqrt{(\alpha^2+2c\alpha+c_1)(\alpha^2-\lambda^2)}}{(\alpha^2-\lambda^2)^2}\,d\alpha, \\ \phi(v) &= -\frac{(\lambda^2-1)\sqrt{\beta^2-\lambda^2+1}}{\beta}\int\frac{\sqrt{(\beta^2+2c\beta+c_1)(\beta^2-\lambda^2+1)}}{(\beta^2-\lambda^2+1)^2}\,d\beta, \end{aligned}\right\} \text{(IV)}$$

where
$$\alpha = \frac{\lambda}{\sqrt{1-u^2}}, \quad \beta = \frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}},$$

and c, c_1 are arbitrary constants.

Surfaces of the Third Family.

These are defined as the envelope of the plane

$$2ux + 2vy + (1 - u^2 - v^2)z = f(u) + \phi(v). \tag{11}$$

The planes of the lines of curvature are

$$x - uz = \frac{f'(u)}{2}, \quad y - vz = \frac{\phi'(v)}{2}. \tag{21}$$

Reasoning in a manner analogous to that used with respect to the other two families, we find that f and ϕ must satisfy the identity

$$\frac{\frac{1}{2} f''(u) + z}{\frac{1}{2}\phi''(v) + z} = \frac{U}{V}, \tag{22}$$

which, after replacing z by its expression as found from (11) and (21), is

$$\frac{\frac{1}{2}(1 + u^2 + v^2) f''(u) + f(u) - uf'(u) + \phi(v) - v\phi'(v)}{\frac{1}{2}(1 + u^2 + v^2)\phi''(v) + f(u) - uf'(u) + \phi(v) - v\phi'(v)} = \frac{U}{V}. \tag{22'}$$

Again we remark that this is the same identity which Caronnet* has found in seeking for those surfaces of the third family which are *isothermic.* As before, we will simply recall his results; they are

$$\left.\begin{aligned} f(u) &= \lambda u^2 + mu + n, \\ \phi(v) &= \lambda' v^2 + m'v + n', \end{aligned}\right\} \tag{I}$$

where λ, m, n, λ', m', n' are arbitrary constants. These surfaces are *cyclides of Dupin* of the *third* order.

$$\left.\begin{aligned} f(u) &= -(3u^2 + u^4), \\ \phi(v) &= 3v^2 + v^4. \end{aligned}\right\} \tag{II}$$

This is the *minimal surface of Enneper,* whose rectangular cartesian coordinates have the following expressions:

$$\begin{aligned} x &= 3u + 3uv^2 - u^3, \\ y &= 3v + 3vu^2 - v^3, \\ z &= 3u^2 - 3v^2. \end{aligned}$$

$$\left.\begin{aligned} f(u) &= (1 - c)^{-\frac{3}{2}} u \tan^{-1} \frac{u}{\sqrt{1 - c}} + \frac{1}{1 - c}, \\ \phi(v) &= c^{-\frac{3}{2}} v \tan^{-1} \frac{v}{\sqrt{c}} + \frac{1}{c}, \end{aligned}\right\} \tag{III}$$

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* These, p. 22.

where c is a constant.

$$\left.\begin{aligned} f(u) &= u \tan^{-1} u + 1, \\ \phi(v) &= \frac{1}{3v^2}. \end{aligned}\right\} \tag{IV}$$

$$\left.\begin{aligned} f(u) &= u \int \frac{\sqrt{(1+u^2)^2 + 2c(1+u^2) + c_1}}{u^2}\, du, \\ \phi(v) &= v \int \frac{\sqrt{v^4 - 2cv^2 + c_1}}{v_2}\, dv, \end{aligned}\right\} \tag{V}$$

where c and c_1 are constants.

It is of interest to remark that the condition (22) can be readily obtained from the general differential equation of the fourth order which ξ satisfies. For, on writing

$$2u = -(\alpha + \beta), \quad 2v = -i(\beta - \alpha), \quad \xi = f(u) + \phi(v),$$

equation (11) takes the form

$$(\alpha + \beta)x + i(\beta - \alpha)y + (\alpha\beta + 1)z + \xi = 0.$$

We have at once

$$p = \frac{\partial \xi}{\partial \alpha} = -\tfrac{1}{2}(f' - i\phi'), \quad q = \frac{\partial \xi}{\partial \beta} = -\tfrac{1}{2}(f' + i\phi'),$$

$$r = \frac{\partial^2 \xi}{\partial \alpha^2} = \tfrac{1}{4}(f'' - \phi''), \quad s = \frac{\partial^2 \xi}{\partial \alpha\, \partial \beta} = \tfrac{1}{4}(f'' + \phi''), \quad t = \frac{\partial^2 \xi}{\partial \beta^2} = \tfrac{1}{4}(f'' - \phi''),$$

hence
$$r = t, \quad s + r = \tfrac{1}{2} f'', \quad s - r = \tfrac{1}{2}\phi''.$$

Substituting these expressions in the equation for ξ, we find that f and ϕ must satisfy the equation

$$\frac{\partial^2}{\partial \alpha^2} \log \left(\frac{\frac{1}{2} f'' + z}{\frac{1}{2}\phi'' + z}\right)^{\frac{1}{2}} - \frac{\partial^2}{\partial \beta^2} \log \left(\frac{\frac{1}{2} f'' + z}{\frac{1}{2}\phi'' + z}\right)^{\frac{1}{2}} = 0,$$

or, by returning to variables u, v,

$$\frac{\partial^2}{\partial u\, \partial v} \log \left(\frac{\frac{1}{2} f'' + z}{\frac{1}{2}\phi'' + z}\right)^{\frac{1}{2}} = 0,$$

that is,
$$\frac{\frac{1}{2} f''(u) + z}{\frac{1}{2}\phi''(v) + z} = \frac{U}{V}.$$

We have shown that when the lines of curvature of a surface are isothermal and isothermal-conjugate, their spherical representation forms an isothermal

system. Further, it is well known that the surface obtained from an isothermic surface by reciprocal radii vectores is isothermic. Hence, combining the results of the preceding sections with those known for isothermic surfaces, we have the result:

Quadrics, the sphere, minimal surfaces, surfaces of revolution, the cyclides of Dupin, the members of the second and third families of surfaces with plane lines of curvature in both systems whose coordinates have been given, together with the surfaces obtained from the preceding by reciprocal radii vectores, have an isothermal spherical representation of their lines of curvature.

PRINCETON UNIVERSITY, *April*, 1901.

Some Differential Equations Connected with Hypersurfaces.

By G. O. James.

§1. Surfaces in ordinary space may be treated from two essentially different standpoints. Either we may suppose a surface given by an equation between the cartesian coordinates of a point on it and deduce its properties by the methods of analytic geometry, or we may suppose the infinite system of which it is a member given by a binary differential quadratic form, the linear element, and deduce its properties by the methods of differential geometry. Similarly, curved spaces of any number of dimensions immersed in homaloidal space may be studied from either standpoint. To any algebraic equation connecting the cartesian coordinates of a point in three dimensional Euclidian space, there corresponds a surface, but in order that there may exist a system of surfaces admitting a binary differential quadratic form as linear element, the coefficients of this form must satisfy certain algebraic and differential relations, namely, the discriminant must be positive, i. e. the form itself must be *definite*, and the equations of Gauss and Codazzi must be satisfied. In the same way, in Euclidian space of any number of dimensions, a manifold of points may be separated from the space by an algebraic equation satisfied by the coordinates of a point in this space, but given an n-ary differential quadratic form, its coefficients must satisfy certain relations in order that there may exist a corresponding manifold admitting this as linear element.

These relations have been investigated from a purely algebraic standpoint by Ricci,* who solves the following problem :

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* Ricci, "Principii di una teoria delle forme differenziale quadratiche," Annale di Mat., Serie II, Vol. 12.

To determine the conditions that the n-ary differential quadratic form

$$f = \sum_1^n {}^{rs}\, a_{rs}\, dx_r\, dx_s$$

shall be reducible, i. e. can be put identically in the form

$$\sum_1^{n-1} {}^{lm}\, b_{lm}\, du_l\, du_m\,,$$

where u_l is a function of $x_1 \ldots . x_n$ and b_{lm} is a function of $u_1 \ldots . u_{n-1}$. When f is irreducible, Schlaefli* has shown that it can be deduced from

$$ds^2 = \sum_1^{n+h} {}^{i}\, dy_i^2\,,$$

where

$$0 \leqq h \leqq \frac{n(n-1)}{2}\,.$$

Ricci defines h as the class of f and investigates the conditions under which f shall be of a given class. Now, since the linear element of the curved space, or manifoldness of n dimensions, is irreducible and of the first class, the problem comes algebraically to the discussion of the conditions under which f shall be of the first class, and Ricci has completely solved this. The object of this paper is to treat the problem as a problem of differential geometry and not as a part of the theory of quadratic forms. For this I shall suppose the curved space given by an algebraic equation between the cartesian coordinates of a point on it and shall deduce the conditions which the coefficients of the first and second fundamental forms must satisfy, and the differential equations on the integration of which the effective determination of the hypersurface depends. From these is derived the theorem of Beez, that a curved space of dimensions greater than two cannot be deformed so as to preserve its linear element, and hence is only capable of translation and rotation in hyperspace if *inextensible.*

To render the analytic work more manageable, I shall confine myself to four dimensions, and shall adopt the nomenclature of Poincaré in his memoir,† "Sur les Residus des Integrales Doubles." Four dimensional homaloidal or Euclidian space will then be termed *hyperspace*, and a single relation between the coordinates of a point in hyperspace will define a *hypersurface.*

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* Annale di Mat., Serie II, Vol. 5, p. 190.

† Acta Math., t. 9, p. 325.

§2. Suppose the hypersurface to be given in the form

$$F(y_1 y_2 y_3 y_4) = 0$$

of linear element

$$f = ds^2 = \Sigma\, dy^2.$$

Expressing the orthogonal coordinates in terms of three independent parameters, x_1, x_2, x_3, by the functions

$$y_i = y_i(x_1 x_2 x_3), \qquad i = 1, 2, 3, 4,$$

which, together with their first, second and third partial derivatives, are supposed uniform finite and continuous throughout the region of variation of x_1, x_2 and x_3, the linear element takes the form

$$f = ds^2 = \sum_1^3{}^{rs} E_{rs}\, dx_r\, dx_s,$$

where

$$E_{rs} = \sum_1^4{}^{i} \frac{\partial y_i}{\partial x_r} \frac{\partial y_i}{\partial x_s}.$$

After the analogy of the theory of surfaces, I shall call this ternary differential quadratic form the *first fundamental form* of the hypersurface.

§3. For a *real* hypersurface, the discriminant of f cannot vanish in general,

$$\Delta = |E_{rs}| = \left\| \begin{matrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \frac{\partial y_3}{\partial x_1} & \frac{\partial y_4}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_3}{\partial x_2} & \frac{d y_4}{\partial x_2} \\ \frac{\partial y_1}{\partial x_3} & \frac{\partial y_2}{\partial x_3} & \frac{\partial y_3}{\partial x_3} & \frac{\partial y_4}{\partial x_3} \end{matrix} \right\|^2 = \Sigma \begin{vmatrix} \frac{\partial y_i}{\partial x_1} & \frac{\partial y_j}{\partial x_1} & \frac{\partial y_k}{\partial x_1} \\ \frac{\partial y_i}{\partial x_2} & \frac{\partial y_j}{\partial x_2} & \frac{\partial y_k}{\partial x_2} \\ \frac{\partial y_i}{\partial x_3} & \frac{\partial y_j}{\partial x_3} & \frac{\partial y_k}{\partial x_3} \end{vmatrix}^2,$$

$$(i, j, k = 1, 2, 3, 4, \qquad i \neq j \neq k),$$

and if $\Delta = 0$, the terms of the second member must separately vanish, which necessitates relations among the coordinates other than the equation of the hypersurfaces. Δ is therefore, in general, positive and different from zero, and f is *definite.*

§4. Applying the formula defining the Christoffel symbols of the first kind,

$$\begin{bmatrix} ik \\ l \end{bmatrix} = \tfrac{1}{2}\left(\frac{\partial E_{il}}{\partial x_k} + \frac{\partial E_{kl}}{\partial x_i} - \frac{\partial E_{ik}}{\partial x_l}\right),$$

we have the following eighteen symbols of the first kind of three indices for the case of three parameters:

$$\begin{bmatrix}11\\1\end{bmatrix}=\tfrac{1}{2}\frac{\partial E_{11}}{\partial x_1};\quad \begin{bmatrix}12\\1\end{bmatrix}=\tfrac{1}{2}\frac{\partial E_{11}}{\partial x_2};\quad \begin{bmatrix}13\\1\end{bmatrix}=\tfrac{1}{2}\frac{\partial E_{11}}{\partial x_3};\quad \begin{bmatrix}22\\1\end{bmatrix}=\frac{\partial E_{12}}{\partial x_2}-\tfrac{1}{2}\frac{\partial E_{22}}{\partial x_1};$$

$$\begin{bmatrix}23\\1\end{bmatrix}=\tfrac{1}{2}\left(\frac{\partial E_{12}}{\partial x_3}+\frac{\partial E_{13}}{\partial x_2}-\frac{\partial E_{23}}{\partial x_1}\right);\quad \begin{bmatrix}33\\1\end{bmatrix}=\frac{\partial E_{13}}{\partial x_3}-\tfrac{1}{2}\frac{\partial E_{33}}{\partial x_3};$$

$$\begin{bmatrix}11\\2\end{bmatrix}=\frac{\partial E_{12}}{\partial x_1}-\tfrac{1}{2}\frac{\partial E_{11}}{\partial x_2};\quad \begin{bmatrix}12\\2\end{bmatrix}=\tfrac{1}{2}\frac{\partial E_{22}}{\partial x_1};\quad \begin{bmatrix}13\\2\end{bmatrix}=\tfrac{1}{2}\left(\frac{\partial E_{12}}{\partial x_3}+\frac{\partial E_{23}}{\partial x_1}-\frac{\partial E_{13}}{\partial x_2}\right);$$

$$\begin{bmatrix}22\\2\end{bmatrix}=\tfrac{1}{2}\frac{\partial E_{22}}{\partial x_2};\quad \begin{bmatrix}23\\2\end{bmatrix}=\tfrac{1}{2}\frac{\partial E_{22}}{\partial x_3};\quad \begin{bmatrix}33\\2\end{bmatrix}=\frac{\partial E_{23}}{\partial x_3}-\tfrac{1}{2}\frac{\partial E_{33}}{\partial x_2};$$

$$\begin{bmatrix}11\\3\end{bmatrix}=\frac{\partial E_{13}}{\partial x_1}-\tfrac{1}{2}\frac{\partial E_{11}}{\partial x_3};\quad \begin{bmatrix}12\\3\end{bmatrix}=\tfrac{1}{2}\left(\frac{\partial E_{13}}{\partial x_2}+\frac{\partial E_{23}}{\partial x_1}-\frac{\partial E_{12}}{\partial x_3}\right);\quad \begin{bmatrix}13\\3\end{bmatrix}=\tfrac{1}{2}\frac{\partial E_{33}}{\partial x_1};$$

$$\begin{bmatrix}22\\3\end{bmatrix}=\frac{\partial E_{23}}{\partial x_2}-\tfrac{1}{2}\frac{\partial E_{22}}{\partial x_3};\quad \begin{bmatrix}23\\3\end{bmatrix}=\tfrac{1}{2}\frac{\partial E_{33}}{\partial x_2};\quad \begin{bmatrix}33\\3\end{bmatrix}=\tfrac{1}{2}\frac{\partial E_{33}}{\partial x_3}.$$

The symbols of the second kind of three indices for a triply orthogonal system of parametric lines will be useful. Making

$$E_{12}=E_{13}=E_{23}=0$$

in the symbols of the first kind, these are given at once, and to get those of the second kind it is sufficient to observe that

$$A_{\nu\nu}=\frac{1}{E_{\nu\nu}},\quad A_{\nu\lambda}=0,\qquad\qquad \nu\neq\lambda,$$

where A_{ij} is the algebraic complement of a_{ij} divided by a, and $E_{rs}=a_{rs}$. Applying the formulæ

$$\begin{Bmatrix}i\varkappa\\\nu\end{Bmatrix}=\sum A_{\nu l}\begin{bmatrix}i\varkappa\\l\end{bmatrix},$$

we have

$$\begin{Bmatrix}11\\1\end{Bmatrix}=\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{11}}}{\partial x_1};\quad \begin{Bmatrix}12\\1\end{Bmatrix}=\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{11}}}{\partial x_2};\quad \begin{Bmatrix}13\\1\end{Bmatrix}=\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{11}}}{\partial x_3};$$

$$\begin{Bmatrix}22\\1\end{Bmatrix}=\frac{\sqrt{E_{22}}}{E_{11}}\frac{\partial\sqrt{E_{22}}}{\partial x_1};\quad \begin{Bmatrix}23\\1\end{Bmatrix}=0;\quad \begin{Bmatrix}33\\1\end{Bmatrix}=-\frac{\sqrt{E_{33}}}{E_{11}}\frac{\partial\sqrt{E_{33}}}{\partial x_3};$$

$$\begin{Bmatrix}11\\2\end{Bmatrix}=-\frac{\sqrt{E_{11}}}{E_{22}}\frac{\partial\sqrt{E_{11}}}{\partial x_2};\quad \begin{Bmatrix}12\\2\end{Bmatrix}=\frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{22}}}{\partial x_1};\quad \begin{Bmatrix}13\\2\end{Bmatrix}=0;$$

$$\begin{Bmatrix}22\\2\end{Bmatrix}=\frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{22}}}{\partial x_2};\quad \begin{Bmatrix}23\\2\end{Bmatrix}=\frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{22}}}{\partial x_3};\quad \begin{Bmatrix}33\\2\end{Bmatrix}=-\frac{\sqrt{E_{33}}}{E_{22}}\frac{\partial\sqrt{E_{33}}}{\partial x_2};$$

$$\begin{Bmatrix}11\\3\end{Bmatrix}=-\frac{\sqrt{E_{11}}}{E_{33}}\frac{\partial\sqrt{E_{11}}}{\partial x_3};\quad \begin{Bmatrix}12\\3\end{Bmatrix}=0;\quad \begin{Bmatrix}13\\3\end{Bmatrix}=\frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{33}}}{\partial x_2};$$

$$\begin{Bmatrix}22\\3\end{Bmatrix}=-\frac{\sqrt{E_{22}}}{E_{33}}\frac{\partial\sqrt{E_{22}}}{\partial x_3};\quad \begin{Bmatrix}23\\3\end{Bmatrix}=\frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{33}}}{\partial x_2};\quad \begin{Bmatrix}33\\3\end{Bmatrix}=\frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{33}}}{\partial x_2}.$$

§5. The *tangent hyperplane* is determined by any three lines not in the same plane lying in it and passing through a point. In particular, it is determined by the three coordinate lines. Defining then the normal to the hypersurface at any point as the line perpendicular to the tangent hyperplane, we have to express the fact that the line whose direction cosines are

$$Y_1,\ Y_2,\ Y_3,\ Y_4,$$

is orthogonal to the coordinate lines. We have then to determine these direction cosines the three equations

$$\sum_i Y_i \frac{\partial y_i}{\partial x_r} = 0, \qquad r = 1, 2, 3.$$

These give at once

$$Y_l = \frac{1}{\sqrt{\Delta}} \frac{\partial (y_m y_p y_q)}{\partial (x_1 x_2 x_3)}, \qquad l \neq m \neq p \neq q, \tag{1}$$

where Δ is the discriminant of the linear element expressed in the E_{rs}. Introducing the second ternary differential quadratic form

$$\begin{aligned} \phi &= -\sum dY dy \\ &= \sum_{rs} D_{rs}\, dx_r\, dx_s, \end{aligned} \tag{2}$$

we at once find

$$D_{rs} = \sum Y \frac{\partial^2 y}{\partial x_r \partial x_s} = -\sum \frac{\partial Y}{\partial x_r} \frac{\partial y}{\partial x_s} = -\sum \frac{\partial Y}{\partial x_s} \frac{\partial y}{\partial x_r}. \tag{3}$$

by means of (1) these can be written

$$D_{rs} = \frac{1}{\sqrt{\Delta}} \begin{vmatrix} \frac{\partial^2 y_1}{\partial x_r \partial x_s} & \frac{\partial^2 y_2}{\partial x_r \partial x_s} & \frac{\partial^2 y_3}{\partial x_r \partial x_s} & \frac{\partial^2 y_4}{\partial x_r \partial x_s} \\ \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \frac{\partial y_3}{\partial x_1} & \frac{\partial y_4}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_4}{\partial x_2} \\ \frac{\partial y_1}{\partial x_3} & \frac{\partial y_2}{\partial x_3} & \frac{\partial y_3}{\partial x_3} & \frac{\partial y_4}{\partial x_3} \end{vmatrix}. \tag{3'}$$

If, now, A_1, A_2, A_3, A_4 be any four functions of x_1, x_2, x_3, then α_1, α_2, α_3, α_4 can be so determined that the four equations

$$A_i = \sum \alpha_r \frac{\partial y_i}{\partial x_r} + \alpha_4 Y_i, \qquad i = 1, 2, 3, 4 \tag{a}$$

are satisfied since the determinant of the system equals $\sqrt{\Delta}$ and is therefore not zero. By exactly the same reasoning as that employed in the case of two parameters,* we arrive at the following system of equations satisfied by the coordinates

$$\frac{\partial^2 y_i}{\partial x_r \partial x_s} = \sum_t \begin{Bmatrix} rs \\ t \end{Bmatrix} \frac{\partial y_i}{\partial x_t} + D_{rs} Y_i, \tag{I}$$

and the following system satisfied by the direction cosines of the normal:

$$\frac{\partial Y_i}{\partial x_r} = -\frac{1}{\Delta} \sum_s \left| D_{1r}^{(s)} \right| \frac{\partial y_i}{\partial x_s}, \tag{II}$$

where

$$|D_{1r}^{(s)}| = \begin{vmatrix} E_{1,\,s-1} & D_{1r} & E_{1,\,s+1} \\ E_{2,\,s-1} & D_{2r} & E_{2,\,s+1} \\ E_{3,\,s-1} & D_{3r} & E_{3,\,s+1} \end{vmatrix}$$

The relations connecting the coefficients of the two forms are now found by writing the conditions of integrability of (I) and (II). These are for (I)

$$\frac{\partial}{\partial x_t}\left(\frac{\partial^2 y_i}{\partial x_r \partial x_s}\right) = \frac{\partial}{\partial x_r}\left(\frac{\partial^2 y_i}{\partial x_s \partial x_t}\right),$$

which give

$$\sum_1^3{}_p \left[\begin{Bmatrix} rs \\ p \end{Bmatrix} \frac{\partial^2 y_i}{\partial x_p \partial x_t} + \frac{\partial}{\partial x_t} \begin{Bmatrix} rs \\ p \end{Bmatrix} \frac{\partial y_i}{\partial x_p} + Y_i \frac{\partial D_{rs}}{\partial x_r} + D_{rs} \frac{\partial Y_i}{\partial x_t} \right]$$

$$= \sum_1^3{}_p \left[\begin{Bmatrix} st \\ p \end{Bmatrix} \frac{\partial^2 y_i}{\partial x_p \partial x_r} + \frac{\partial}{\partial x_r} \begin{Bmatrix} st \\ p \end{Bmatrix} \frac{\partial y_i}{\partial x_p} + Y_i \frac{\partial D_{st}}{\partial x_r} + D_{st} \frac{\partial y_i}{\partial x_r} \right].$$

Substituting for $\frac{\partial^2 y_i}{\partial x_r \partial x_s}$ and $\frac{\partial Y_i}{\partial x_t}$ from (I) and (II) we have

$$\sum_1^3{}_{pq} \left[\left(\begin{Bmatrix} rs \\ p \end{Bmatrix} \begin{Bmatrix} pt \\ q \end{Bmatrix} - \begin{Bmatrix} st \\ p \end{Bmatrix} \begin{Bmatrix} pr \\ q \end{Bmatrix} + \frac{\partial}{\partial x_t} \begin{Bmatrix} rs \\ q \end{Bmatrix} - \frac{\partial}{\partial x_r} \begin{Bmatrix} st \\ q \end{Bmatrix} - \frac{D_{rs}}{\Delta} \left| D_{1t}^{(q)} \right| + \frac{D_{st}}{\Delta} \left| D_{1r}^{(q)} \right| \right) \frac{\partial y_i}{\partial x_q} \right.$$

$$\left. + \left(\frac{\partial D_{rs}}{\partial x_t} - \frac{\partial D_{st}}{\partial x_r} + D_{qt} \begin{Bmatrix} rs \\ q \end{Bmatrix} - D_{qr} \begin{Bmatrix} st \\ q \end{Bmatrix} \right) Y_i \right] = 0. \tag{b}$$

This system is again of type (a) with the first member zero, and hence the coefficients must separately vanish. Introducing the symbol of four indices

$$\{ps,\, rt\} = \frac{\partial}{\partial x_t} \begin{Bmatrix} rs \\ p \end{Bmatrix} - \frac{\partial}{\partial x_r} \begin{Bmatrix} st \\ p \end{Bmatrix} + \sum_q \left(\begin{Bmatrix} rs \\ p \end{Bmatrix} \begin{Bmatrix} pt \\ q \end{Bmatrix} - \begin{Bmatrix} st \\ p \end{Bmatrix} \begin{Bmatrix} pr \\ q \end{Bmatrix} \right),$$

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* Bianchi, Lezioni, p. 87.

we have the systems

$$\left\{sq, rt\right\} - \frac{D_{rs}}{\Delta}\left|D_{1t}^{(q)}\right| + \frac{D_{st}}{\Delta}\left|D_{1r}^{(q)}\right| = 0, \tag{III}$$

$$\frac{\partial D_{rs}}{\partial x_t} - \frac{\partial D_{st}}{\partial x_r} + \sum_p \left(D_{pt}\begin{Bmatrix} rs \\ p \end{Bmatrix} - D_{pr}\begin{Bmatrix} st \\ p \end{Bmatrix}\right) = 0, \tag{IV}$$

$$(p, q, r, s, t = 1, 2, 3).$$

These equations can be written in a different form.

In the identities

$$(s\,p\,r\,t) = \sum_q a_{qp}\,\{s\,q\,r\,t\},$$

substituting for $\{s\,q\,r\,t\}$ from (III), we have at once, after reduction,

$$(s\,p\,r\,t) = D_{rs}\,D_{pt} - D_{st}\,D_{pr}. \tag{III'}$$

Since six only of the symbols $(s\,p\,r\,t)$ are independent and different from zero, the coefficients of the second fundamental form are completely determined, the eight relations (IV), which may be looked upon as the generalization* of the Mainardi-Codazzi equations.

From (IV) we have at once

$$\frac{\partial D_{lp}}{\partial x_m} - \frac{\partial D_{lm}}{\partial x_p} + D_{lm}\begin{Bmatrix} lp \\ l \end{Bmatrix} - D_{lp}\begin{Bmatrix} lm \\ l \end{Bmatrix} + \sum' \left(D_{rm}\begin{Bmatrix} lp \\ r \end{Bmatrix} - D_{rp}\begin{Bmatrix} lm \\ r \end{Bmatrix}\right) = 0, \tag{c}$$

where, in $\sum'$, $r = l$ is excluded.

Observing that

$$\frac{\partial \log \sqrt{a}}{\partial x_l} = \sum_{i\kappa} A_{i\kappa}\begin{bmatrix} il \\ \kappa \end{bmatrix}$$

we have

$$\frac{\partial\left(\frac{1}{\sqrt{a}}\right)}{\partial x_l} = \sum_i \frac{1}{\sqrt{a}}\begin{Bmatrix} il \\ i \end{Bmatrix}.$$

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* Ricci, Acc. dei L. Rend. 2, 1895, §V, p. 320; Cesáro, "Geometria Intrinseca," p. 238.

Hence

$$D_{lp}\frac{\partial\left(\frac{1}{\sqrt{\Delta}}\right)}{\partial x_m}-D_{lm}\frac{\partial\left(\frac{1}{\sqrt{\Delta}}\right)}{\partial x_p}+\frac{1}{\sqrt{\Delta}}\Big[\Big(D_{lp}\left\{\begin{matrix}lm\\ l\end{matrix}\right\}-D_{lm}\left\{\begin{matrix}lp\\ l\end{matrix}\right\}\Big)$$

$$+\sum'\Big(D_{lp}\left\{\begin{matrix}rm\\ r\end{matrix}\right\}-D_{lm}\left\{\begin{matrix}rp\\ r\end{matrix}\right\}\Big)\Big]=0. \quad \text{(d)}$$

Adding (c) and (d),

$$\frac{\partial\left(\frac{D_{lp}}{\sqrt{\Delta}}\right)}{\partial x_m}-\frac{\partial\left(\frac{D_{lm}}{\sqrt{\Delta}}\right)}{\partial x_p}+\frac{1}{\sqrt{\Delta}}\Big[\sum'\Big(D_{rm}\left\{\begin{matrix}lp\\ r\end{matrix}\right\}-D_{rp}\left\{\begin{matrix}lm\\ r\end{matrix}\right\}$$

$$+D_{lp}\left\{\begin{matrix}rm\\ r\end{matrix}\right\}-D_{lm}\left\{\begin{matrix}rp\\ r\end{matrix}\right\}\Big)\Big]=0. \quad \text{(IV}')$$

Since the D_{rs} are immediately expressible in terms of the E_{rs} and their derivatives by (III′), we have here eight differential relations satisfied by the coefficients of the linear element of the hypersurface. Hence we may say, *in order that the* DEFINITE *ternary differential quadratic form*

$$f=\sum_{rs} a_{rs}\,dx_r\,dx_s$$

shall represent the linear element of a hypersurface, the coefficients must satisfy (IV').

§6. *Conversely, given the ternary differential quadratic form*

$$f=\sum_{rs} a_{rs}\,dx_r\,dx_s,$$

which is DEFINITE *and whose coefficients satisfy* (IV'), *there exists a* UNIQUE *hypersurface admitting f as linear element, and in order to effectively obtain this hypersurface, it is necessary to integrate three generalized Riccati equations.*

Suppose the hypersurface, whose existence and uniqueness we wish to determine under the hypotheses above, to be referred to its *lines of curvature*, and consider at every point the *tetrarectangular tetrahedroid* formed by the tangents to the positive directions of these three lines and the normal. Let $Y_{\lambda\mu}$ ($\lambda=1, 2, 3, 4$) be the direction cosines of the tangent to the line x_μ and $Y_{\lambda 4}$, those of the normal. Then

$$Y_{\lambda u}=\frac{1}{\sqrt{E_{\mu\mu}}}\,\frac{\partial y_\lambda}{\partial x_\mu},$$

$$Y_{\lambda 4}=Y_\lambda.$$

From (I) and (II), p. 254, substituting for the Christoffel symbols their values from p. 252, we have

$$\frac{\partial Y_{11}}{\partial x_1} = -\frac{Y_{12}}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial x_2} - \frac{Y_{13}}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial x_3} + \frac{D_{11}}{\sqrt{E_{11}}}Y_{14};$$

$$\frac{\partial Y_{11}}{\partial x_2} = \frac{Y_{12}}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial x_1}; \quad \frac{\partial Y_{11}}{\partial x_3} = \frac{Y_{13}}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial x_1};$$

$$\frac{\partial Y_{12}}{\partial x_1} = \frac{Y_{11}}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial x_2}; \quad \frac{\partial Y_{12}}{\partial x_2} = \frac{Y_{13}}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial x_2};$$

$$\frac{\partial Y_{12}}{\partial x_3} = -\frac{Y_{11}}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial x_1} - \frac{Y_{13}}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial x_3} + \frac{D_{22}}{\sqrt{E_{22}}}Y_{14};$$

$$\frac{\partial Y_{13}}{\partial x_1} = \frac{Y_{11}}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial x_3}; \quad \frac{\partial Y_{13}}{\partial x_2} = \frac{Y_{12}}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial x_3};$$

$$\frac{\partial Y_{13}}{\partial x_3} = -\frac{Y_{11}}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial x_1} - \frac{Y_{12}}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial x_2} + \frac{D_{33}}{\sqrt{E_{33}}}Y_{14};$$

$$\frac{\partial Y_{14}}{\partial x_1} = -\frac{D_{11}}{\sqrt{E_{11}}}Y_{11}; \quad \frac{\partial Y_{14}}{\partial x_2} = \frac{-D_{22}}{\sqrt{E_{22}}}Y_{12}; \quad \frac{\partial Y_{14}}{\partial x_3} = \frac{-D_{33}}{\sqrt{E_{33}}}Y_{13}.$$

The four functions $Y_{\lambda\mu}$ $(\mu = 1, 2, 3, 4)$ then satisfy the four simultaneous linear homogeneous total differential equations

$$\left.\begin{aligned}
dY_{\lambda 1} &= \left\{-\frac{Y_{\lambda 2}}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial x_2} - \frac{Y_{\lambda 3}}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial x_3} + \frac{D_{11}}{\sqrt{E_{11}}}Y_{\lambda 4}\right\}dx_1 \\
&\qquad + \frac{Y_{\lambda 2}}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial x_1}dx_2 + \frac{Y_{\lambda 3}}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial x_1}dx_3, \\
dY_{\lambda 2} &= \frac{Y_1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial x_2}dx_1 \\
&\qquad + \left\{-\frac{Y_{\lambda 1}}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial x_1} - \frac{Y_{\lambda 3}}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial x_3} + \frac{D_{22}}{\sqrt{E_{22}}}Y_{\lambda 4}\right\}dx_2 \\
&\qquad + \frac{Y_{\lambda 3}}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial x_2}dx_3, \\
dY_{\lambda 3} &= \frac{Y_{\lambda 1}}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial x_3}dx_1 + \frac{Y_{\lambda 2}}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial x_3}dx_2 \\
&\qquad + \left\{-\frac{Y_{\lambda 1}}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial x_1} - \frac{Y_{\lambda 2}}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial x_2} + \frac{D_{33}}{\sqrt{E_{33}}}Y_{\lambda 4}\right\}dx_3, \\
dY_{\lambda 4} &= -\frac{D_{11}}{\sqrt{E_{11}}}Y_{\lambda 1}dx_1 - \frac{D_{22}}{\sqrt{E_{22}}}Y_{\lambda 2}dx_2 - \frac{D_{33}}{\sqrt{E_{33}}}Y_{\lambda 3}dx_3.
\end{aligned}\right\} \quad (4)$$

This is an illimitably integrable system in virtue of equations (III) and (IV),

remembering that in the case of the hypersurface referred to its lines of curvature,

$$E_{rs} = D_{rs} = 0, \qquad r \neq s.$$

Hence there exists an integral system, and a single one, which, for the initial values of the variables $x_i = x_i^0$, reduces to arbitrarily given initial values.

If $Y_{\lambda\mu}$ and $Y'_{\lambda\mu}$ are two integral systems, then

$$\sum_1^4{}^{\mu}\, Y_{\lambda\mu}\, Y'_{\lambda\mu} = \text{const.}$$

Let $Y_{\lambda\mu}$ ($\lambda, \mu = 1, 2, 3, 4$) be four integral systems of (4) which for initial values $x_i = x_i^0$, reduce to the sixteen coefficients of an orthogonal substitution. Then it follows from the observation above, that for all values of the variables will these sixteen quantities be the coefficients of an orthogonal substitution, and in particular,

$$\sum_1^4{}^{\lambda}\, Y_{\lambda\mu}^2 = 1,$$

$$\sum_1^4{}^{\lambda}\, Y_{\lambda\mu}\, Y_{\lambda\nu} = 0, \qquad (\mu \neq \nu).$$

From (4) it is easily seen that the four expressions

$$\sum_1^3{}^{i} \sqrt{E_{ii}}\, Y_{\lambda i}\, dx_i, \qquad (\lambda = 1, 2, 3, 4)$$

are exact differentials, and writing

$$y_\lambda = \int \sum_1^3{}^{i} \sqrt{E_{ii}}\, Y_{\lambda i}\, dx_i,$$

we have a hypersurface with the given fundamental forms.

§7. The system (4) is identical with systems (34), (34′), (34″) found by Professor Craig* when $\alpha, \beta, \gamma, \delta$ are replaced by $Y_{\lambda 1}, Y_{\lambda 2}, Y_{\lambda 3}, Y_{\lambda 4}$ and the p_{ij} by the coefficients in (4). Now Professor Craig has shown that the integration of (34) can be reduced to the integration of a generalized† Riccati equation

* Amer. Jour., Vol. XX, No. 2, p. 145. † L. c., p. 141.

and in the same way (4) can be reduced to the integration of the three generalized Riccati equations

$$\left.\begin{aligned}
\frac{\partial\lambda}{\partial x_1} &= -\frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial x_2}\mu - \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial x_3}\nu + \frac{\varkappa^2-1}{2}\cdot\frac{D_{11}}{\sqrt{E_{11}}} - \lambda^2\frac{D_{11}}{\sqrt{E_{11}}}, \\
\frac{\partial\mu}{\partial x_1} &= \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial x_2}\lambda - \lambda\mu\frac{D_{11}}{\sqrt{E_{11}}}, \\
\frac{\partial\nu}{\partial x_1} &= \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial x_3}\lambda - \lambda\nu\frac{D_{11}}{\sqrt{E_{11}}},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\frac{\partial\lambda}{\partial x_2} &= \frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial x_2}\mu - \mu\lambda\frac{D_{22}}{\sqrt{E_{22}}}, \\
\frac{\partial\mu}{\partial x_2} &= -\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial x_1}\lambda - \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial x_3}\nu + \frac{\varkappa^2-1}{2}\frac{D_{22}}{\sqrt{E_{22}}} - \mu^2\frac{D_{22}}{\sqrt{E_{22}}}, \\
\frac{\partial\nu}{\partial x_2} &= \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial x_3}\mu - \mu\nu\frac{D_{22}}{\sqrt{E_{22}}},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\frac{\partial\lambda}{\partial x_3} &= \frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial x_1}\nu - \nu\lambda\frac{D_{33}}{\sqrt{E_{33}}}, \\
\frac{\partial\mu}{\partial x_3} &= \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial x_2}\nu - \nu\mu\frac{D_{33}}{\sqrt{E_{33}}}, \\
\frac{\partial\nu}{\partial x_3} &= -\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial x_1}\lambda - \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial x_2}\mu + \frac{\varkappa^2-1}{2}\frac{D_{33}}{\sqrt{E_{33}}} - \nu^2\frac{D_{33}}{\sqrt{E_{33}}}.
\end{aligned}\right\}$$

By the substitutions

$$Y_{\lambda 1} = \frac{2\lambda}{\varkappa^2+1}; \quad Y_{\lambda 2} = \frac{2\mu}{\varkappa^2+1}; \quad Y_{\lambda 3} = \frac{2\nu}{\varkappa^2+1}; \quad Y_{\lambda 4} = \frac{\varkappa^2-1}{\varkappa^2+1};$$
$$\varkappa^2 = \lambda^2 + \mu^2 + \nu^2.$$

§7. Here we have chosen the special tetrahedroid formed by the tangents to the lines of curvature and the normal, but any other *tetrarectangular tetrahedroid* might have been taken and we should have arrived at a set of equations similar to (4), illimitably integrable in virtue of equations (III) and (IV), and these would have led to three generalized Riccati equations of the form

$$\left.\begin{aligned}
\frac{\partial\lambda}{\partial x_i} &= a_{1i}\mu + b_{1i}\nu + \frac{\varkappa^2-1}{2}c_{1i} + \lambda\,(a'_{1i}\lambda + b'_{1i}\mu + c'_{1i}\nu), \\
\frac{\partial\mu}{\partial x_i} &= a_{2i}\lambda + b_{2i}\nu + \frac{\varkappa^2-1}{2}c_{2i} + \mu\,(a'_{2i}\lambda + b'_{2i}\nu + c'_{2i}\nu), \\
\frac{\partial\nu}{\partial x_i} &= a_{3i}\lambda + b_{3i}\mu + \frac{\varkappa^2-1}{2}c_{3i} + \nu\,(a'_{3i}\lambda + b'_{3i}\mu + c'_{3i}\nu).
\end{aligned}\right\}$$

§8. It is interesting to note that the equations (III) and (IV), which must be satisfied in order that

$$f = \sum_{1}^{3}{}_{rs} E_{rs}\, dx_r\, dx_s$$

shall represent the linear element of a hypersurface, are exactly the conditions that f shall be irreducible and of the first class.* From the definition of D_{rs} we observe that

$$D_{rs} = (rs),$$

and equations (III′), p. 255, become exactly equations (I), p. 153, and substituting $D_{rs} = (rs)$ in equations (IV), p. 255, we have (II), p. 153.

§9. From equations (III′), p. 255, it follows that the second fundamental form is completely determined when the first is given, and from the theorem on p 256, it follows that in this case the hypersurface is *uniquely* determined to within motion in hyperspace. Hence the property possessed by surfaces in Euclidian space of three dimensions of being deformed without alteration of the linear element, cannot be extended to hypersurfaces. We thus come upon a theorem noted by Beez,† and further put in evidence by Ricci.‡

* Ricci, "Principii di una teoria," etc., p. 151.
† Cesáro, "Lezioni di Geometria Intrinseca," p. 247.
‡ "Principii di una teoria," etc., p. 163.

BALTIMORE, MD., 1901.

On the Forms of Sextic Scrolls of Genus Greater than One.

By Virgil Snyder.

The following paper is a direct continuation of my previous ones on the Sextic Scrolls which appeared in the Journal, Vol. XXV, pp. 58–96. The scroll is generated by the lines joining corresponding points of two directrix curves lying upon it.

§1.—*Sextic Scrolls of Genus* 2.

1. In this case the nodal curve is of order 8 and the simplest plane curve existing on the surface is a nodal quartic. Given two nodal quartics having the same characteristics which lie in different planes but have two points of intersection. These two curves can be put in (1, 1) point correspondence in such a way that the two points of intersection are self-corresponding elements. Lines joining corresponding points will generate a sextic scroll of genus 2. This is the general case, type I.

Consider the curve

$$z = 0, \quad y^2 = \frac{f_4(x)}{f_2(x)}.$$

Make any Cremona transformation upon it which will produce a quartic curve. The lines joining corresponding points can be rationally expressed in terms of hyper-elliptic functions.

In case the line of intersection of the two planes is a double generator, every plane passed through it will cut from the surface a nodal quartic having the same characteristics. The residual nodal curve of order 7 must therefore cut the double generator in six points. A plane containing the double generator and a single generator must accordingly contain either a double directrix and a single generator, or a triple directrix, hence the residual curve of order 7 must be composite. A triple generator cannot exist.

2. Consider a straight line and a nodal quartic. Put them in (2, 1) correspondence by making the points on the line projective with the pencil through the node. Connect these points with the points in which the line cuts the quartic again.

The equations of the line are

$$x = 0, \quad y = 0, \quad z = \mu,$$

and those of a variable generator connecting the point $(0, 0, \mu)$ with $(x_1, y_1, 0)$ are

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{\mu - z}{\mu},$$

wherein

$$y_1^2 = \frac{f_4(x_1)}{f_2(x_1)} \quad \text{and} \quad \mu = \frac{Ax_1 + B}{Cx_1 + D}.$$

If $\qquad f_4(x) \equiv \alpha' x^4 + \beta' x^3 + \gamma' x^2 + \delta' x + \varepsilon', \quad f_2(x) \equiv \alpha x^2 + \beta x + \gamma,$

the equation may be written in the form

$$\begin{vmatrix} x^2\alpha' - y^2\alpha & x^2\beta' - y^2\beta & x^2\gamma' - y^2\gamma & \delta' x^2 & \varepsilon' x^2 & 0 \\ 0 & x^2\alpha' - y^2\alpha & x^2\beta' - y^2\beta & x^2\gamma' - y^2\gamma & \delta' x^2 & \varepsilon' x \\ Cz - Aw & Ax + Dz - Bw & Bx & 0 & 0 & 0 \\ 0 & Cz - Aw & Ax + Dz - Bw & Bxw & 0 & 0 \\ 0 & 0 & Cz - Aw & Ax + Dz - Bw & Bx & 0 \\ 0 & 0 & 0 & Cz - Aw & Ax + Dz - Bw & Bw \end{vmatrix} = 0.$$

which contains the extraneous factor x.

The line x, y is a double directrix; the residual nodal curve of order 7 cuts the directrix once and every generator three times. Type II.

If the origin lies on the curve $\varepsilon' = 0$ and the double directrix is also a simple generator, the residual nodal curve is a quintic which cuts the multiple line and every other generator twice. Type III.

If, in the general case, $B = 0$, the equation may be written

$$\begin{aligned} &\varepsilon' x^2 (Cz - Aw)^4 - (Cz - Dw)^3 (Ax + Dz)\, \delta' x^2 \\ &\quad + (Cz - Aw)^2 (Ax + Dz)^2 (x^2\gamma - y^2\gamma') - (Cz - Aw)^3 (Ax + Dz)^3 (x^2\beta' - y^2\beta) \\ &\quad + (Ax + Dz)^4 (x^2\alpha' - y^2\alpha) = 0, \end{aligned}$$

which shows that the line $Ax + Dz = 0$, $Cz - Aw = 0$ is a four-fold directrix. The line $x = 0$, $z = 0$ is a double generator. This is the general (2, 4) corre-

spondence with one double element between the points of two skew lines. Type IV.

When the directrix line passes through the node and also the vertex of the pencil, the two directrices become coincident. By a slight change of coordinates the equation may be written

$$xf_3(x, w)(xy - azw) + (xy - azw)^2 f_2(x, w) = x^2 f_4(x, w).$$

The line $x = 0$, $z = 0$ is a double generator as before, but the four-fold directrix is now $x = 0$, $w = 0$. Type V.

3. Let $f_4(x)$ be of the form $x(x - \varkappa)\phi_2(x)$, and let the pencil of lines through $x = \varkappa$, $y = 0$ be projective with the points of $x = 0$, $y = 0$ in such a way that the line of the pencil which passes through the origin corresponds to the origin. By using μ in the same sense as before, the equation of the pencil is

$$y = \frac{A\mu(x - \varkappa)}{C\mu + D},$$

from which the equation of the scroll generated by lines connecting the point $(0, 0, \mu)$ with the points in which the corresponding line of the pencil cuts the quartic again is

$$y\,[Dy - A\varkappa z]\,f_2(Dxy - A\varkappa zx,\ Czy + Dyw - Axz)$$
$$= x(Dx - \varkappa(Cz + Dw))\,\phi_2(Dxy - A\varkappa xz,\ Czy + Dyw - Axz).$$

The line $x = 0$, $y = 0$ is a triple directrix; another triple directrix is $Dy - A\varkappa z = 0$, $Dx - \varkappa(Cz + Dw) = 0$. There are two double generators, $x = 0$, $Cz + Dw$ and $y = 0$, $z = 0$.

This is the general (3, 3) correspondence with two double elements between the points of two skew lines. Type VI.

In order to obtain the corresponding surface when the directrices are coincident, let (l, m) be any point on the curve. The pencil

$$(y - m) = \nu(x - l)$$

is to be made projective with the points of the line $x = l$, $y = m$ in such a way that the point $z = 0$ corresponds to the tangent

$$[f_3^2(l) - f_4'(l)](x - l) + 2mf_2(l)(y - m) = 0.$$

The subsequent procedure is the same as before. Type VII.

If, in VI, x be put directly equal to zero, the equation reduces to type V. The reason is that the tangent to the curve at the origin passes through the node, thus imposing an extra condition.

A particular case of VII may be generated by letting m and l each become zero, but letting the pencil be defined by

$$x = \frac{A\mu y}{B\mu + C}.$$

The resulting equation is

$$y^2 f_2(Cx^2,\ Cxw + Bzx - Azy) = f_3(Cx^2,\ Cxw + Bzx - Azy),$$

in which $xf_3 = f_4$ when $\varepsilon' = 0$.

The line $z = 0$, $x = 0$ now counts for two consecutive double generators. Any plane section not containing the triple directrix nor this line will cut the latter in a tacnode of the curve of section.

Types IV to VII are the only sextic scrolls of genus 2 that are contained in a linear congruence.*

4. A surface containing a triple directrix but not contained in a linear congruence, may be generated as follows: Consider a pencil of conics passing through the node and three other basis points on the quartic. Each conic will cut the quartic in three further points. Make the pencil projective with the points of a line which cuts the quartic in one point—the correspondence being such that the point of intersection of the line and the quartic corresponds to the conic through the same point. Lines joining points of the fixed directrix line to the points in which the associated conic cuts the quartic will generate the scroll. The residual nodal curve is a quintic which cuts the directrix and every generator twice. Type VIII. The difference between types III and VIII lies in the configuration of the directrix. In III, two generators distinct from the directrix issue from each point of it; in VIII, there are three such generators and the directrix is not itself a generator.

If the quartic curve has a cusp, the line joining it to the corresponding point of the directrix will be a double generator. The residual is now a quartic cutting the directrix once and every generator twice. Type IX.

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* The most general quintic scroll having a tacnodal generator can be expressed in the form

$$y^3(yw - zx) = x(xy - a(yw - zx))^2.$$

It is unicursal.

In case the directrix in the general form (VIII) passes through the node, it becomes a triple line and simple generator. The residual nodal curve is a conic section cutting the directrix in one point. Type X.

Finally, let the line go through the node, but have the conic pass through four simple basis points on the quartic. As before, the node, regarded as a point on the directrix, is to correspond to the conic through it. The line is now a four-fold directrix. The residual is a conic which cuts it once. Type XI.

The configuration of nodal curves in X and XI suggests using these curves for directrices, the former by a (3, 2) correspondence with the point of intersection as a simple point, and the latter has the point of intersection for a double element.

Let
$$\mu^2 f_3(\lambda) + \mu\phi_3(\lambda) + \lambda f_2(\lambda) = 0,$$
wherein $\mu = \dfrac{yz}{yw - x^2}$, $\lambda = \dfrac{y}{x}$. The scroll is expressed by
$$yz^2 f_3(x, y) + z(yw - x^2)\phi_3(x, y) + (yw - x^2)^2 f_2(x, y) = 0.$$
Similarly, for the last type the correspondence is
$$\mu^2 f_4(\lambda) + \mu\lambda f_3(\lambda) + \lambda^2 f_2(\lambda) = 0,$$
from which
$$z^2 f_4(x, y) + z(yw - x^2) f_3(x, y) + (yw - x^2)^2 f_2(x, y) = 0.$$
The more particular case of a double directrix and a double generator is impossible for a surface of genus 2, since it would establish a (2, 2) correspondence between unicursal curves. The resulting scroll is given among those of genus 1 as type XII.

When the genus of a sextic scroll is greater than 1, the scroll cannot have a triple conic nor more thon one double conic as part of the nodal curve.

§2.—*Forms of Sextic Scrolls of Genus* 2.

I.	c_8^2.	VII.	$d^3 \equiv d'^3 + 2g^2$.
II.	$d^2 + c_7^2$.	VIII.	$d^3 + c_5^2$.
III.	$(d^2, g^1) + c_5^2$.	IX.	$d^3 + c_4^2 + g^2$.
IV.	$d^4 + d^2 + g^2$.	X.	$(d^3, g^1) + c_2^2$.
V.	$(d^4 \equiv d^2, g^2) + g^2$.	XI.	$d^4 + c_2^2$.
VI.	$2d^3 + 2g^2$.		

§3—*Sextic Scrolls of Genus* 3.

5. The nodal curve is now of order 7; the simplest curve lying on the surface is a non-singular quartic. Given two non-singular quartics having the same characteristics. Let them be in different planes but have two points of intersection. Establish a (1, 1) correspondence between the points of the two curves and connect corresponding points by straight lines. This scroll will be the general type of genus 3. The nodal curve is of order 7. Type I.

A double generator cannot exist without making the residual curve composite, neither can a double directrix for the latter would necessitate a (1, 2) correspondence between the points of a non-singular quartic and a straight line which intersects it, which is impossible. The nodal curve cannot consist of a double conic and a simple or composite nodal quintic, for if a plane be passed through the two generators which intersect at any point on the conic, the plane would cut the residual curve in six points, three on each generator.

The only remaining forms are accordingly those contained in a linear congruence.

6. Put a quartic curve and a line which intersects it in one point in (1, 3) correspondence by making the points of the line projective with a pencil whose vertex is any point on the quartic. Let the line of the pencil which cuts the directrix correspond to the point in which it cuts the directrix. The line will be a triple directrix. A double generator will exist in the plane of the quartic and the residual curve will be another triple directrix skew to the first and passing through the vertex of the pencil.

The general equation may be written in the form

$$y^3 f_3(z, w) + y^2 x \phi_3(z, w) + y x^2 z f_2(z, w) + x^3 z^2 f_1(z, w) = 0,$$

in which the triple directrices are $x = 0$, $y = 0$; $z = 0$, $w = 0$, and the double directrix is $y = 0$, $z = 0$. Type II.

When the vertex of the pencil lies on the directrix, the two directrices become coincident. The equation now is

$$y^2 f_4(x, y) + y f_3(x, y)(xz - wy) + f_2(x, y)(xz - wy)^2 + x(xz - wy)^3 = 0. \quad \text{Type III.}$$

7. The preceding are the only sextic scrolls of genus 2 on which quartic curves can lie. Two more forms exist which can be generated by replacing the

quartic by a quintic having a triple point at the vertex of the pencil. The directrix is now double and the residual is a four-fold directrix passing through the triple point. The equation is

$$y^2 f_4(z, w) + xy\phi_4(z, w) + x^2\psi_4(z, w) = 0. \qquad \text{Type IV.}$$

When the first directrix line also passes through the triple point, the two directrices become coincident. The equation is

$$f_6(x, y) + f_4(x, y)(xz - wy) + f_2(x, y)(xz - yw)^2 = 0. \qquad \text{Type V.}$$

§4.—*Nodal Curves of Sextic Scrolls of Genus* 3.

I. c_7^2. II. $2d^3 + g^2$. IV. $d^4 + d^2$.
III. $d^3 \equiv d'' + g^2$. V. $d^4 \equiv (d^2, g^2)$.

§5.—*Sextic Scrolls of Genus* 4.

8. Since any plane section must be of genus 4, it follows that any plane section containing two generators can only contain generators or directrix lines. Since the most general (2, 4) correspondence between the points of two lines generates a scroll of genus 3, it follows that the (3, 3) correspondence is the only one possible.

Given a quintic curve with two double points. Let a directrix line passing through one of them be made projective with a pencil lying in the plane of the quintic and vertex at the other node, the line joining the two nodes corresponding to the node on the directrix.

The equation of the resulting scroll is

$$y^3 f_3(z, w) + y^2 x\phi_3(z, w) + yx^2\psi_3(z, w) + x^3\chi_3(z, w) = 0. \qquad \text{Type I.}$$

When the two nodes become consecutive, forming a tacnode, the two directrices become coincident. The equation now is

$$f_4(x, y) + f_4(x, y)(zx - yw) + f_2(x, y)(zx - yw)^2 + \kappa(zx - yw)^3 = 0. \qquad \text{Type II.}$$

9. These are the only forms of sextic scrolls of genus 4:

I. $2d^3$. II. $d^3 \equiv d''$.

10. In general, the highest genus that a scroll of order $2n$ can have is $(n-1)^2$. The nodal curve consists of two skew or coincident n-fold directrix lines. The simplest plane curve that can exist on the surface is one of order $2n-1$ having two $(n-1)$-fold points. A scroll of order $2n$ and having $(n-1)^2$ double generators exists, and is unicursal.

Similarly, if the order of the scroll be $2n-1$, the highest genus is $n(n-1)$. The orders of the two directrices are $n, n-1$. The order of the simplest plane curve is $2n-2$ and it has an $(n-1)$-fold and another $(n-2)$-fold point. A unicursal scroll of order $2n-1$ may have $n(n-1)$ double generators.

CORNELL UNIVERSITY, *August*, 1902.

Geometry on the Cuspidal Cubic Cone.

BY FREDERICK C. FERRY.

The geometry on the general cubic surface has attracted the attention of many mathematicians, and papers on the subject have been published by *Clebsch, †Diekmann, ‡Sturm, §Rohn, and others. The geometry on the cubic cone seems to have received by itself little consideration, which may be due to the fact that no great difficulty is found in applying to these particular surfaces the results obtained for the general cubic.

The following paper has for its object to investigate by the methods of analytic geometry the question of the order of the surface of lowest order which can be passed through any given proper curve on the cuspidal cubic cone, and also the character of the residual intersection in each case.

A system of coördinates, λ, μ, ν, is selected for the representation by homogeneous equations of curves on the cone in question, the degree of any such equation in λ and μ being denoted by p and its degree in ν by q. Any curve, whose equation is so represented, is found to pass $p-q$ times through the vertex of the cone, and to meet every element of the cone in q points apart from that vertex. The order of the curve as a curve in space is proved to be $p+2q$; and this order is congruent, modulus 3, to the order of multiplicity of the vertex of the cone as a point of the curve.

The investigation of the order of the surface S of lowest order (save in general the given cone itself), which can contain the curve considered, is then taken up, and it is demonstrated that that order is as low as p in every case, while the

* Kronecker J., LXV, 359-380

† Math. Annalen, IV, 442-475.

‡ Math. Annalen, XXI, 457-515.

§ Leipz. Ber., XLVI, 84-119.

residual intersection is made up entirely of the cuspidal edge of the given cone occurring $2(p-q)$ times; but, if $p=q$, the curve is one of total intersection. If, however, $p>q+2$, the order of this surface S can be made lower than p by means of the substitution of x^2z for y^3 in the equation of S, which renders that equation factorable by a power of x; thus it is proved that this order of S can be so far reduced in every case that the residual intersection becomes of order not greater than four; and the residual intersection is still made up entirely of the cuspidal edge of the given cone occurring the requisite number of times.

It is then pointed out that a curve given by an equation of entirely general form in the coördinates chosen passes $2q$ times through the infinite point of the cuspidal edge of the given cone, and meets that edge nowhere else except at the vertex of the cone; accordingly, the special form of equation necessary for the representation of curves which meet that cuspidal edge elsewhere than at the two points mentioned, and the number of conditions imposed in each case are determined. Then the order of the surface S of lowest order containing the curves of this kind is ascertained; and it is proved that three of the four classes of curves here occurring are curves of total intersection, while in the case of each curve of the remaining class the residual intersection is made up entirely of the cuspidal edge of the given cone found twice only.

A table of results for the curves of the lower orders is given at the end of the paper.

The methods used throughout the paper are analogous to those employed by the same writer in papers on the geometry on the cubic scrolls.*

I.—*The Cuspidal Cubic Cone.*

The equation of the cuspidal cubic cone in homogeneous coördinates is

$$y^3-x^2z=0.$$

Let this cone, or the left member of its equation, as the case may demand, be denoted by Σ. The equation $x=0$ gives the cuspidal tangent plane, which has contact with the cone all along the cuspidal edge and accordingly contains that

* "Geometry on the Cubic Scroll of the First Kind," Archiv for Mathematik og Videnskab, B. XXI, Nr. 3, p. 3-57. "Geometry on the Cubic Scroll of the Second Kind," Amer. Journ. Math., XXIII, p. 179-235.

edge three times; $y = 0$ gives the plane through both the cuspidal and the inflectional edges, containing the former edge twice and the latter edge once; $z = 0$ gives the inflectional tangent plane, which has contact with Σ all along the inflectional edge and hence contains that edge three times; and $s = 0$ gives the infinite plane, whose intersection with Σ is a cuspidal cubic curve, the "infinite cubic" in the geometry under consideration. The cuspidal edge of Σ is given by the equations $x = 0$, $y = 0$, and the inflectional edge by $y = 0$, $z = 0$; while $z = 0$, $s = 0$ are the equations of the inflectional tangent to the infinite cubic, $s = 0$, $x = 0$ give the cuspidal tangent to the infinite cubic, $y = 0$, $s = 0$ give a line through the cusp and the point of inflection of the infinite cubic, hence an infinite line, and $x = 0$, $z = 0$ give a line through the vertex of Σ but containing no other point of the cone. The vertex of Σ is given by the equations $x = 0$, $y = 0$, $z = 0$, the cusp of the infinite cubic by $x = 0$, $y = 0$, $s = 0$, and the point of inflection of the infinite cubic by $y = 0$, $z = 0$, $s = 0$; the point whose equations are $x = 0$, $z = 0$, $s = 0$ does not lie on Σ.

II.—*Coördinates on* Σ.

Taking for coördinates on Σ the variables λ, μ and ν, so chosen that $x/y = \lambda/\mu$ and $x/s = \lambda/\nu$, the equation $\Sigma = 0$ gives the following relation between the coördinates of the two systems, viz.

$$x : y : z : s = \lambda^3 : \lambda^2\mu : \mu^3 : \lambda^2\nu.$$

Then $\lambda = 0$ gives the cuspidal edge of Σ, $\mu = 0$ gives the inflectional edge of Σ, and $\nu = 0$ gives the intersection of Σ with the infinite plane, i. e., the infinite cubic. The vertex of Σ is given by the equations $\lambda = 0$, $\mu = 0$, the cusp of the infinite cubic by $\lambda = 0$, $\nu = 0$, and the point of inflection of the infinite cubic by $\mu = 0$, $\nu = 0$. Any point of the cuspidal edge of Σ is given by an equation of the form $\mu = k\nu$, where k is a constant; thus, if k have the value zero, the equation gives the vertex of Σ, and, if k have the value infinity, the equation gives the cusp of the infinite cubic. Similarly, any point of the inflectional edge is given by an equation of the form $\lambda = k\nu$; this equation gives the vertex of Σ if k has the value zero, and the point of inflection of the infinite cubic if k has the value infinity And, in like manner, any point of the infinite cubic is determined by an equation of the form $\lambda = k\mu$; the point thus determined will be the cusp of the infinite cubic if k has the value zero, and the point of inflection of that cubic if k becomes infinite.

III.—*Curves on* Σ.

It is evident that any proper curve on Σ may be represented by an irreducible homogeneous equation in the coördinates λ, μ and ν. Let such an equation be designated by $\phi = 0$, and let the degrees of that equation in all three variables and in the variable ν respectively be denoted by p and q, which demands that $p \geqq q$. The equation in question may then be arranged according to powers of ν and represented thus:

$$\phi \equiv \phi_p + \nu \cdot \phi_{p-1} + \nu^2 \cdot \phi_{p-2} + \cdots\cdot + \nu^\theta \cdot \phi_{p-\theta} + \cdots\cdot + \nu^q \cdot \phi_{p-q} = 0,$$

and the curve given by this equation may be called *the curve* (p, q).

Since the equation $\phi = 0$ is of degree $p - q$ in the variables λ, μ, it is evident that the curve (p, q) in question has $p - q$ points coincident at the point on Σ given by the equations $\lambda = 0$, $\mu = 0$, i, e., *the curve* (p, q) *passes* $p - q$ *times through the vertex of* Σ. These $p - q$ points at the vertex of Σ will be regarded as lying on the $p - q$ edges of Σ given by the equation $\phi_{p-q} = 0$, and not, in general, on the cuspidal edge of Σ; if, however, ϕ_{p-q} has λ^{a_1} as a factor, then will a_1 of the $p - q$ points in question be regarded as lying on the cuspidal edge.

The cuspidal edge of Σ is given by the equation $\lambda = 0$; if the value zero be inserted in place of λ in the equation $\phi = 0$, there results in general an equation in μ, ν of degree p of the form $\mu^{p-q} . \psi_q = 0$, where ψ_q is homogeneous in μ, ν; the factor μ^{p-q} corresponds to the $p - q$ points of the curve at the vertex of Σ, while $\psi_q = 0$ gives the q values of μ/ν for the q remaining points where the curve meets the cuspidal edge of Σ; hence, *the curve* (p, q) *meets the cuspidal edge of* Σ *in* q *points in addition to the* $p - q$ *points of the curve at the vertex of* Σ.

Similarly, the inflectional edge of Σ is given by the equation $\mu = 0$, and, on inserting that value for μ in the equation $\phi = 0$, there results in general an equation of the form $\lambda^{p-q} . \psi'_q = 0$, where ψ'_q is homogeneous in λ, ν; the factor λ^{p-q} corresponds to the $p - q$ points of the curve at the vertex of Σ, while $\psi'_q = 0$ gives the q values of λ/ν for the q remaining points where the curve (p, q) meets the inflectional edge of Σ; hence, *the curve* (p, q) *meets the inflectional edge of* Σ *in* q *points in addition to the* $p - q$ *points of the curve at the vertex of* Σ. In the general case μ cannot be a factor of ψ_q and λ cannot be a factor of ψ'_q; the special cases where such factors occur will be considered later.

Any plane through the cuspidal and inflectional edges of Σ will pass through the vertex of Σ and will contain the cuspidal edge twice and the inflectional edge once. Consequently, such a plane will contain $p-q$ points of the curve (p, q) at the vertex, $2q$ points of that curve along the cuspidal edge, and q points along the inflectional edge; thus this plane will contain $p-q+2q+q=p+2q$ points of the curve in question. Hence, in the general case, *the order of the curve* (p, q) *as a curve in space is* $p+2q$. Let m denote this order of the curve (p, q); then, in general, $m=p+2q$.

An edge or element of Σ is a straight line, hence of order unity as a curve in space; consequently, an edge of Σ is a curve (1, 0), and as such is given by an equation of the form $a\lambda+b\mu=0$. This equation gives the cuspidal edge if b/a has the value zero, and the inflectional edge if b/a is infinite. Since this equation of the edge does not involve the variable ν, it follows that an edge, in general, has $q=0$ points, i. e., no points in common with the cuspidal edge or the inflectional edge, apart from the vertex of Σ. Thus, *a homogeneous equation* $\phi_p(\lambda, \mu)=0$ *of degree p in the coördinates λ, μ represents p edges of* Σ. These p edges may be wholly or in part distinct or coincident, and all together may be regarded as making up a $(p, 0)$, as their equation $\phi_p=0$ shows. Evidently, the only curve having $q=0$ is that consisting of an edge or group of edges of Σ.

To determine the points of intersection of any edge, whose equation is $a\lambda+b\mu=0$, with the curve (p, q) given by $\phi=0$, it is necessary to give λ the value $\lambda=-b/a.\mu$ in the equation $\phi=0$; this equation then takes the form $\mu^{p-q}.\psi_q''=0$, where ψ_q'' is a homogeneous polynomial in μ, ν of degree q; the factor μ^{p-q} corresponds to the $p-q$ points of the curve at the vertex of Σ, and $\psi_q''=0$ gives q values of μ/ν corresponding to the q points where the curve meets the edge in question. Hence, *the curve* (p, q) *has q points on each edge of* Σ, *in addition to the $p-q$ points of the curve lying at the vertex of* Σ. Particular edges of Σ, as already implied, may have one or more of their q points of the curve lying at the vertex of Σ, but none of the $p-q$ points at the vertex of Σ is to be regarded as lying on more than a single one of the edges at that point.

Since $p+2q$ is congruent to $p-q$, modulus 3, it follows that the order of the curve (p, q) as a space curve is congruent, modulus 3, to the number of times the curve in question passes through the vertex of Σ; if π denote the order of multiplicity of the vertex of Σ as a point of the curve (p, q), then *m and π are connected by the relation* $m\equiv\pi$ (mod 3).

Two curves (p, q) and (p', q'), where $p = p'$ and $q = q'$, are said to be of the same *species*. The theorem stated above shows that the only possible curves, when $\pi \equiv 0 \pmod 3$, are of the orders 3, 6, 9, 12, 15, etc., and of the species (1, 1), (2, 2), (3, 3), (3, 0), (4, 1), (4, 4), etc.; if $\pi \equiv 1 \pmod 3$, the only possible curves are of the orders 1, 4, 7, 10, 13, 16, etc., and of the species (1, 0), (2, 1), (3, 2), (4, 0), (4, 3), (5, 1), etc., and, if $\pi \equiv 2 \pmod 3$, the only possible curves are of the orders 2, 5, 8, 11, 14, 17, etc., and of the species (2, 0), (3, 1), (4, 2), (5, 0), (5, 3), (6, 1), etc.

It will be supposed, unless otherwise stated, that the equation $\phi = 0$ is irreducible and entirely general in form; consequently, that equation will not represent henceforth any curve $(p, 0)$ where p has a value greater than unity.

IV.—*The Curve (p, q) as the Intersection, Total or Partial, of a Surface S with Σ.*

To find the equation of a surface S cutting the curve (p, q) from Σ, it is necessary to substitute x, y, z and s in the equation of the curve (p, q), according to the laws connecting the space-coördinates x, y, z, s with the Σ-coördinates λ, μ, ν; i. e., λ^3, $\lambda^2\mu$, μ^3 and $\lambda^2\nu$ are to be replaced respectively by x, y, z and s in accordance with the proportion

$$x : y : z : s = \lambda^3 : \lambda^2\mu : \mu^3 : \lambda^2\nu.$$

Or, the same result may be accomplished by replacing λ, μ and ν respectively by x, y and s, which substitutions satisfy the proportion

$$x : y : z : s = \lambda : \mu : \mu^2/\lambda^2 . \mu : \nu.$$

Following the latter substitutions, any curve (p, q), given by the equation $\phi = 0$, meaning $\phi(\lambda, \mu, \nu) = 0$, is cut from Σ by a surface whose equation is $\phi(x, y, s) = 0$; the surface is, in this case, a cone with its vertex at the point given by $x = 0$, $y = 0$, $s = 0$ or $\lambda = 0$, $\nu = 0$, and will be designated by C. Since ϕ is of degree $p - q$ in λ, μ, the equation $\phi(x, y, s) = 0$ is of degree $p - q$ in the variables x, y; hence the cuspidal edge of Σ occurs $p - q$ times on C as an edge of that cone, corresponding to the occurrence of the vertex of Σ as a $(p - q)$-tuple point of the curve (p, q). The cuspidal edge of Σ accordingly counts $2(p - q)$ times in the intersection of Σ and C, leaving a remainder of order $3p - 2(p - q) = p + 2q = m$, which remainder is the curve (p, q). Hence, *the curve (p, q) is cut from Σ by a cone C of order p whose residual intersection with Σ is made up entirely of the cuspidal edge of Σ occurring $2(p - q)$ times.*

Let the order of the residual intersection in any case be denoted by R, while R_i shall denote that portion of the order of the residual intersection due to the intersection of sheets of surfaces regardless of contact of sheets, and R_c shall denote the remainder of the order of the residual intersection, the portion of the whole order of the residual intersection which arises solely from contact between the surfaces in question; then, evidently, $R = R_i + R_c$. Let the order of multiplicity of the cuspidal edge of Σ on the other surface under consideration be designated by f.

A sufficient condition in order that the curve (p, q) be the complete intersection of Σ and C is that R have the value zero, demanding that $2(p-q)=0$ or $p=q$; such a curve does not pass at all through the vertex of Σ and is characterized by that fact among all the curves (p, q) given by equations of entirely general form. Thus, *any curve* (p, q) *having* $p=q$ *is the complete intersection of* Σ *with a cone* C *of order* p. Evidently no curve of this variety can be cut from Σ by a surface of order less than p. Curves of this kind will be grouped in the class designated as Class I.

The vertex of the cone C is the infinite point of the cuspidal edge of Σ,—the cusp of the infinite cubic,—a point having a special relation to the geometry under consideration; and the question arises as to the number of times any curve (p, q) passes through this point. In order to determine this, the intersections of the curve (p, q) with the infinite cubic will be considered. The infinite cubic is given by the equation $\nu = 0$; hence, if the equation of the curve have its terms arranged according to the powers of ν, thus

$$\phi \equiv \phi_p + \nu \, . \, \phi_{p-1} + \, \cdots \, + \nu^\theta . \, \phi_{p-\theta} + \, \cdots \, + \nu^q . \, \phi_{p-q} = 0,$$

the insertion of the value zero for the variable ν gives the equation $\phi_p = 0$, homogeneous in λ, μ, determining the p edges of Σ on which p intersections of the curve (p, q) with the infinite cubic lie; or, $\phi_p = 0$ gives p edges of Σ, on which edges infinite points of the curve in question are found; or, again, the equation $\phi_p = 0$ determines p directions in each of which the curve (p, q) meets the infinite plane. In the general case λ will not be a factor of ϕ_p, and, consequently, none of these p points will lie at the cusp of the infinite cubic. From the form of the equation $\phi(x, y, s) = 0$, which gives the cone C, it is evident not only that the cone C contains the cuspidal edge of Σ $p-q$ times, but also that the point in question is of multiplicity of order q on C in addition to the

multiplicity due to the multiple edge; this leads to the occurrence of the vertex of the cone C as a $2q$-tuple point in the intersection of Σ with C, in addition to the occurrence of the cuspidal edge of Σ $2(p-q)$ times in that intersection. Hence, *the curve* (p, q) *passes* $2q$ *times through the cusp of the infinite cubic.* This makes the infinite plane meet the curve (p, q) in these $2q$ points, besides the p points elsewhere on the infinite cubic, giving in all $p+2q$ points of intersection, —the order m of the curve in question. The only curves (p, q), given by equations of general form, which do not pass at all through the infinite point of the cuspidal edge of Σ, are, then, those having $q=0$, i. e., edges of the cone Σ; other curves (p, q) having this property, but given by equations of special form, will be considered later.

Any curve (p, q) has then, in general, $p-q+2q$ or $p+q$ points at the vertices of the two cones Σ and C; and, since a plane through these two points contains the cuspidal and some other edge of Σ, the latter edge containing q points of (p, q), it follows that the cuspidal edge of Σ contains, in general, no points of the curve (p, q) apart from those at the vertices of Σ and C. Hence, *the equation* $\phi=0$, *in its most general form, can represent only those special curves* (p, q) *which cut the cuspidal edge of* Σ *at its infinite point and at the vertex of* Σ or at either of these two points. Thus the curves of Class I can meet the cuspidal edge of Σ only at the cusp of the infinite cubic, and must each have a $2p$-tuple point there. The particular forms of the equation $\phi=0$, which give the most general irreducible curves on Σ, are for the present neglected.

Let ρ denote the number of times the curve (p, q) passes through the cusp of the infinite cubic; then, in the general case, as has been seen, $\rho=2q$.

The cones Σ and C can have no other common edge than the cuspidal edge of Σ, for the residual intersection is entirely made up of the cuspidal edge of Σ; and it is supposed, in general, that $m \geqq 2$. Any plane through the cuspidal edge of Σ contains also a second edge of Σ, and on that second edge lie q points of the curve (p, q); hence, through each one of these q points of (p, q) must pass an edge of the cone C; these q edges will be, in general, distinct, and will contain no other points of (p, q) apart from the vertex of C. If Q denote the number of points in which any edge of C meets the curve (p, q) apart from the vertex of C, and P, after the analogy of p, be defined by the equation $P+2Q=m$, then will it be possible to call the (p, q) on Σ a (P, Q) on C. But it has been seen already that $Q=1$, hence $P=m-2Q=m-2=p+2q-2$. The curve

(P, Q) passes through the vertex of C $2q$ times and through the vertex of Σ $p-q$ times. The cuspidal edge of Σ is a $(p-q)$-tuple edge of C, hence any plane through it contains $p-(p-q)=q$ other edges of C; on each of these q edges lies a point of the curve (P, Q); these q points, together with the $p+q$ points on the $(p-q)$-tuple edge, make up the $p+2q$ points in which the curve (P, Q) or the curve (p, q) meets the plane in question.

Every edge of Σ meets the curve (p, q) $p-q$ times at the vertex of Σ, but only those edges will be regarded as meeting the curve there which are given by the equation $\phi_{p-q}=0$, where ϕ_{p-q} is the coefficient of the highest power of ν occurring in the equation of the curve (p, q); $\phi_{p-q}=0$ is homogeneous of degree $p-q$ in λ, μ and gives the $p-q$ edges of Σ to which the curve (p, q) is tangent at the vertex of Σ, as will be shown in a subsequent section. The curve (p, q) can then have the direction of the cuspidal edge at the vertex of Σ only if ϕ_{p-q} has a factor λ, and the direction of the inflectional edge only when μ is a factor of the same polynomial; these cases have thus far been excluded from consideration.

This completes the discussion of the curve (p, q) as cut from Σ by the cone C, and the single class of curves of complete intersection here found may now be characterized thus:

I. $p=q,\ m=3p,\ \pi=0,\ \rho=2q=2p,\ m'=p,\ f=R=R_i=R_c=0.$

Any term in the equation $\phi(x, y, z)=0$, which contains y^3, may have that factor changed to x^2z as many times as it occurs, since $\Sigma\equiv y^3-x^2z=0$; and the surface S, whose equation is $C\equiv C'+C''y^3=0$, has the same intersection with Σ as the surface whose equation is obtained from $C=0$ by adding any multiple of Σ thus:

$$C-C''\Sigma=C'+C''y^3-C''\Sigma=C'+C''x^2z=0.$$

This substitution of x^2z for y^3, wherever the latter occurs, will give at once the equation of a surface S which is not in general a cone, although it still cuts the curve (p, q) from Σ and has its residual intersection with Σ made up entirely of the cuspidal edge of Σ; and, further, it will make possible in many cases the reduction of the equation of that surface S by a factor x^a, giving a surface, not a

cone, of order less by α than p, i. e., giving a surface which is of the order $m' = p - \alpha$. The residual intersection must consist in this case of the cuspidal edge of Σ occurring to the number of $2(p-q) - 3\alpha$ times. If the factor x^α has $3\alpha = 2(p-q)$, then will the curve (p, q) in question be a curve of complete intersection; this demands that $p - q \equiv \theta$ (mod 3), and thus it is seen that here *a curve of complete intersection passes* $3k$ *times through the vertex of* Σ, k being an integer as great as zero. The form which the equation $\phi = 0$ must have, in order that it be factorable thus after the substitution is performed, is found in the following manner: Let the equation $\phi = 0$ be arranged according to the powers of the variable ν and represented thus:

$$\phi \equiv \phi_p + \nu . \phi_{p-1} + \nu^2 . \phi_{p-2} + \dots + \nu^{q-\gamma} . \phi_{p-q+\gamma} + \dots + \nu^q . \phi_{p-q} = 0,$$

and, further, let the general term of the polynomial $\phi_{p-q+\gamma}$ be denoted by $a_{\beta,\gamma}\lambda^\beta\mu^{p-q+\gamma-\beta}$. Inserting x and y in place of λ and μ respectively in this general term, there results the form $a_{\beta,\gamma}x^\beta y^{p-q+\gamma-\beta}$, which, on substituting x^2z for y^3 as many times as possible, reduces to

$$a_{\beta,\gamma}x^{\frac{1}{3}\beta+\frac{2}{3}(p-q+\gamma)}z^{\frac{1}{3}(p-q+\gamma-\beta)}, \quad \text{if } p-q+\gamma-\beta \equiv 0 \pmod 3,$$

or $$a_{\beta,\gamma}x^{\frac{1}{3}\beta+\frac{2}{3}(p-q+\gamma-1)}yz^{\frac{1}{3}(p-q+\gamma-\beta-1)}, \quad \text{if } p-q+\gamma-\beta \equiv 1 \pmod 3,$$

or $$a_{\beta,\gamma}x^{\frac{1}{3}\beta+\frac{2}{3}(p-q+\gamma-2)}y^2z^{\frac{1}{3}(p-q+\gamma-\beta-2)}, \quad \text{if } p-q+\gamma-\beta \equiv 2 \pmod 3,$$

three cases, according to the congruence of the index of the power of μ to the modulus 3. That a factor x^α, where $\alpha = \frac{2}{3}(p-q)$ be removable by division from each of these terms, it is necessary that

$$\tfrac{1}{3}\beta + \tfrac{2}{3}(p-q+\gamma) \geqq \tfrac{2}{3}(p-q), \text{ i. e., } \beta \geqq -2\gamma, \text{ if } p-q+\gamma-\beta \equiv 0 \pmod 3$$

$$\tfrac{1}{3}\beta + \tfrac{2}{3}(p-q+\gamma-1) \geqq \tfrac{2}{3}(p-q), \text{ i. e., } \beta \geqq 2-2\gamma, \quad \text{if } p-q+\gamma-\beta \equiv 1 \pmod 3,$$

$$\tfrac{1}{3}\beta + \tfrac{2}{3}(p-q+\gamma-2) \geqq \tfrac{2}{3}(p-q), \text{ i. e., } \beta \geqq 4-2\gamma, \quad \text{if } p-q+\gamma-\beta \equiv 2 \pmod 3.$$

If $\gamma = 0$, the last terms of ϕ_{p-q}, on which terms alone of this group any restriction could be placed by these conditions, are of the form

$$a_{3,0}\lambda^3\mu^{p-q-3} + a_{2,0}\lambda^2\mu^{p-q-2} + a_{1,0}\lambda\mu^{p-q-1} + a_{0,0}\mu^{p-q},$$

and, since $p - q \equiv 0$ (mod 3), the necessary conditions for β are all satisfied if only $a_{1,0}$ vanish. Similarly, it is seen that no restrictions whatever are imposed

by these conditions for β on any of the terms in the groups where $\gamma \geqq 1$. Hence, there is given a curve of complete intersection whenever $p-q \equiv 0 \pmod 3$, and $a_{1,0}=0$. This case, which includes the curves of Class I above (the condition $a_{1,0}=0$ being always satisfied there), may be called Class I′, and is characterized thus:

I′. $p>q,\ p-q \equiv 0 \pmod 3,\ a_{1,0}=0,\ m=p+2q,\ \pi=p-q,$
$$\rho=2q,\ m'=\tfrac{1}{3}(p+2q),\ f=R=R_i=R_c=0.$$

In general it is not possible to divide out from the equation of the surfaces S a factor of degree as great as the order of the residual intersection, but a certain reduction can usually be made. The nature of such reduction in the general case will now be considered. The equation $\phi=0$, when arranged according to the ascending powers of ν, takes the form

$$\phi \equiv \phi_p + \nu\cdot\phi_{p-1} + \nu^2\cdot\phi_{p-2} + \cdots\cdots + \nu^\theta\cdot\phi_{p-\theta} + \cdots\cdots + \nu^q\cdot\phi_{p-q} = 0;$$

here arise at once three cases, according as

$$\left.\begin{array}{ll} & 1)\ p-q \equiv 0 \pmod 3, \\ & 2)\ p-q \equiv 1 \pmod 3, \\ \text{or} & 3)\ p-q \equiv 2 \pmod 3; \end{array}\right\} \text{i. e., according as} \left\{\begin{array}{ll} & 1)\ p-q = 3\Delta, \\ & 2)\ p-q = 3\Delta+1, \\ \text{or} & 3)\ p-q = 3\Delta+2. \end{array}\right.$$

where Δ is an integer as great as zero.

Case I. $p-q=3\Delta$. Here Δ may be supposed to be as great as unity, since the case where $p-q=0$ has been already considered. The general term of the polynomial ϕ_{p-q} is of the form $a_r\lambda^{3\Delta-r}\mu^r$; this term becomes at once $a_r x^{3\Delta-r}y^r$, and this, after neglecting the coefficient and changing y^3 to x^2z as many times as possible, may be reduced to one of three forms, according to the congruence of r to the modulus 3, thus:

$$\begin{aligned} x^{3\Delta-r}y^r &= x^{3\Delta-r+\frac{2}{3}r}z^{\frac{1}{3}r} \equiv x^{3\Delta-\frac{1}{3}r}z^{\frac{1}{3}r}, && \text{if } r\equiv 0 \pmod 3; \\ &= x^{3\Delta-r+\frac{2}{3}(r-1)}yz^{\frac{1}{3}(r-1)} \equiv x^{3\Delta-\frac{1}{3}(r+2)}yz^{\frac{1}{3}(r-1)}, && \text{if } r\equiv 1 \pmod 3; \\ &= x^{3\Delta-r+\frac{2}{3}(r-2)}y^2z^{\frac{1}{3}(r-2)} \equiv x^{3\Delta-\frac{1}{3}(r+4)}y^2z^{\frac{1}{3}(r-2)}, && \text{if } r\equiv 2 \pmod 3. \end{aligned}$$

In each case there occurs a factor x^α, where $\alpha \geqq 3\Delta-\frac{1}{3}(r+4) \geqq \frac{2}{3}(p-q)-1$, since $\Delta=\frac{1}{3}(p-q)$ and $r \leqq p-q$; hence division by $x^{\frac{2}{3}(p-q)-1}$ is made possible in all the terms obtained from the group ϕ_{p-q}. Since the terms derived from any other group $\phi_{p-\theta}$, where $\theta \leqq q-1$, will be at least one degree higher in the variables x, y than those coming from ϕ_{p-q}, it is evident that the factor

$x^{\frac{2}{3}(p-q)-1}$ can be rejected from the entire equation. The order of the surface S then becomes $m' = p - \frac{2}{3}(p-q) + 1 = \frac{1}{3}(p+2q) + 1$, and the residual intersection must consist of the cuspidal edge of Σ occurring $2(p-q) - 3\alpha = 2(p-q) - 2(p-q) + 3 = 3$ times, requiring that $f = 1$, $R_i = 2$ and $R_c = 1$; hence the surface S has the plane given by $x = 0$ for its tangent plane all along the cuspidal edge of Σ, the surfaces S and Σ having contact all along that line. The same may be found at once to be true by investigating somewhat more carefully the forms of the terms given above, thus: Dividing out $x^{\alpha} \equiv x^{\frac{2}{3}(p-q)-1}$ from each of the three forms given leaves

$$
\begin{aligned}
& x^{p-q-\frac{1}{3}r-\frac{2}{3}(p-q)+1} z^{\frac{1}{3}r} \equiv x^{\frac{1}{3}(p-q)-\frac{1}{3}r+1} z^{\frac{1}{3}r}, && \text{if } r \equiv 0 \pmod{3}, \\
& x^{p-q-\frac{1}{3}(r+2)-\frac{2}{3}(p-q)+1} yz^{\frac{1}{3}(r-1)} \equiv x^{\frac{1}{3}(p-q)-\frac{1}{3}(r-1)} yz^{\frac{1}{3}(r-1)}, && \text{if } r \equiv 1 \pmod{3}, \\
\text{or}\quad & x^{p-q-\frac{1}{3}(r+4)-\frac{2}{3}(p-q)+1} y^2 z^{\frac{1}{3}(r-2)} \equiv x^{\frac{1}{3}(p-q)-\frac{1}{3}(r+1)} y^2 z^{\frac{1}{3}(r-2)}, && \text{if } r \equiv 2 \pmod{3}.
\end{aligned}
$$

Since $r \leqq p - q$, the lowest terms in x, y come from the first group and are of the first power in those variables, containing only x; all terms from the second and third groups will be of at least the second degree in x, y and will contain y. Curves of the kind here considered will be said to belong to Class II and may be characterized thus:

II. $p > q,\ p - q \equiv 0 \pmod{3},\ m = p + 2q,\ \pi = p - q,\ \rho = 2q,$
$m' = \frac{1}{3}(p + 2q) + 1,\ f = 1,\ R = 3,\ R_i = 2,\ R_c = 1.$

Evidently no curves of complete intersection are included in this class.

Case 2. $p - q = 3\Delta + 1$. The general term of ϕ_{p-q}, of the form $a_r \lambda^{3\Delta - r + 1} \mu^r$, becomes at once $a_r x^{3\Delta - r + 1} y^r$; and this, after neglecting the coefficient and changing y^3 to $x^2 z$ as many times as possible, takes one of three forms,

$$
\begin{aligned}
x^{3\Delta-r+1} y^r &= x^{3\Delta-r+1+\frac{2}{3}r} z^{\frac{1}{3}r} \equiv x^{3\Delta-\frac{1}{3}r+1} z^{\frac{1}{3}r}, && \text{if } r \equiv 0 \pmod{3}, \\
&= x^{3\Delta-r+1+\frac{2}{3}(r-1)} yz^{\frac{1}{3}(r-1)} \equiv x^{3\Delta-\frac{1}{3}(r-1)} yz^{\frac{1}{3}(r-1)}, && \text{if } r \equiv 1 \pmod{3}, \\
&= x^{3\Delta-r+1+\frac{2}{3}(r-2)} y^2 z^{\frac{1}{3}(r-2)} \equiv x^{3\Delta-\frac{1}{3}(r+1)} y^2 z^{\frac{1}{3}(r-1)}, && \text{if } r \equiv 2 \pmod{3}.
\end{aligned}
$$

These terms are all divisible by x^{α}, where $\alpha = 3\Delta - \frac{1}{3}(r+1) \geqq \frac{2}{3}(p - q - 1)$. It is evident that all the terms of higher degrees in x, y coming from the groups $\phi_{p-\theta}$, where $\theta \leqq q - 1$ can be divided by the same x^{α}; hence, the entire equation contains the factor $x^{\frac{2}{3}(p-q-1)}$. The order of the surface S is then $p - \alpha = p - \frac{2}{3}(p - q - 1) = \frac{1}{3}(p + 2q + 2)$, and the residual intersection consists of the cuspidal edge of Σ occurring $2(p-q) - 3\alpha = 2(p-q) - 2(p-q-1) = 2$ times, demanding that $f = 1$, $R = R_i = 2$ and $R_c = 0$; thus the surface S con-

tains the cuspidal edge of Σ once. The same may be seen at once to be true from the forms of the terms given above. Curves of this kind will be grouped in Class III and may be described thus:

III. $p-q\equiv 1 \pmod 3$, $m=p+2q$, $\pi=p-q$, $\rho=2q$,
$m'=\frac{1}{3}(p+2q+2)$, $f=1$, $R=2$, $R_i=2$, $R_c=0$.

Case 3. $p-q=3\Delta+2$. The general term of ϕ_{p-q} is of the form $a_r\lambda^{3\Delta-r+2}\mu^r$; this becomes at once $a_r x^{3\Delta-r+2}y^r$; and this, after neglecting the coefficient a_r and replacing y^3 by x^2z as many times as possible, takes one of three forms, thus:

$$\begin{aligned} x^{3\Delta-r+2}y^r &= x^{3\Delta-r+2+\frac{2}{3}r}z^{\frac{1}{3}r} \equiv x^{3\Delta-\frac{1}{3}r+2}z^{\frac{1}{3}r}, && \text{if } r\equiv 0 \pmod 3, \\ &= x^{3\Delta-r+2+\frac{2}{3}(r-1)}yz^{\frac{1}{3}(r-1)} \equiv x^{3\Delta-\frac{1}{3}(r-4)}yz^{\frac{1}{3}(r-1)}, && \text{if } r\equiv 1 \pmod 3, \\ &= x^{3\Delta-r+2+\frac{2}{3}(r-2)}y^2z^{\frac{1}{3}(r-2)} \equiv x^{3\Delta-\frac{1}{3}(r-2)}y^2z^{\frac{1}{3}(r-2)}, && \text{if } r\equiv 2 \pmod 3). \end{aligned}$$

These terms are all divisible by x^α, where $\alpha=3\Delta-\frac{1}{3}(r-2)\geqq\frac{2}{3}(p-q-2)$. It is evident that all the terms of higher degrees in x, y, coming from the groups $\phi_{p-\theta}$, where $\theta\leqq q-1$, can be divided by the same x^α, hence $x^{\frac{2}{3}(p-q-2)}$ is a factor of the entire equation. The order of the surface S may be reduced thus to $p-\alpha=p-\frac{2}{3}(p-q-2)=\frac{1}{3}(p+2q+4)$, and the residual intersection consists of the cuspidal edge of Σ occurring $2(p-q)-3\alpha=2(p-q)-2(p-q-2)=4$ times, giving $R=4$; and, since each term above is of degree at least as great as two in the variables x, y, after the rejection of the factor x^α, it is clear that $f=2$, $R_i=4$, and, consequently, $R_c=0$; accordingly, the surface S contains the cuspidal edge of Σ twice. This case includes the curves which will be said to form Class IV, and may be characterized thus:

IV. $p-q\equiv 2 \pmod 3$, $m=p+2q$, $\pi=p-q$, $\rho=2q$,
$m'=\frac{1}{3}(p+2q+4)$, $f=2$, $R=4$, $R_i=4$, $R_c=0$.

Since $p-q\equiv p+2q \pmod 3$, and since a necessary condition for a curve (p, q) of complete intersection is that $m=p+2q\equiv 0 \pmod 3$, it follows that no curves of total intersection are included in Classes III and IV.

This disposes of the cases of all curves (p, q) which are represented by equations of the most general form for the values of p and q in question. But, since every such general equation has been found to represent a curve (p, q) passing $p-q$ times through the vertex of Σ and $2q$ times through the cusp of the infinite cubic, it is evident that many general curves, including all curves of

order greater than unity which do not contain either the vertex of Σ or the cusp of the infinite cubic, remain to be given by equations of special forms.

Curves (p, q) given by Equations of Special Forms.

It is possible to determine the conditions under which a higher power of x than that given above can be rejected from the equation obtained from the performance of the required substitutions in the equation $\phi = 0$; such greater reduction will give a surface of lower order containing the curve in question, and it may happen that the curve itself will suffer a reduction in its order by the rejection of some of the terms which appear in the general equation for its arrangement of p and q.

If the terms of the general equation of the curve (p, q) are arranged according to powers of ν, thus:

$$\phi \equiv \phi_p + \phi_{p-1} \cdot \nu + \phi_{p-2} \cdot \nu^2 + \cdots + \phi_{p-q+\gamma} \cdot \nu^{q-\gamma} + \cdots + \phi_{p-q} \cdot \nu^q = 0,$$

any term of the general group $\phi_{p-q+\gamma}$ is of the form $a_{\beta,\gamma}\lambda^\beta\mu^{p-q+\gamma-\beta}$ and, if the substitution be performed, this term becomes at once $a_{\beta,\gamma}x^\beta y^{p-q+\gamma-\beta}$; and this, on changing y^3 to x^2z as many times as possible, takes the form

$$a_{\beta,\gamma}x^{\frac{1}{3}\beta+\frac{2}{3}(p-q+\gamma)}z^{\frac{1}{3}(p-q+\gamma-\beta)}, \quad \text{if } p-q+\gamma-\beta \equiv 0 \pmod 3,$$

or $$a_{\beta,\gamma}x^{\frac{1}{3}\beta+\frac{2}{3}(p-q+\gamma-1)}yz^{\frac{1}{3}(p-q+\gamma-\beta-1)}, \quad \text{if } p-q+\gamma-\beta \equiv 1 \pmod 3,$$

or $$a_{\beta,\gamma}x^{\frac{1}{3}\beta+\frac{2}{3}(p-q+\gamma-2)}y^2z^{\frac{1}{3}(p-q+\gamma-\beta-2)}, \quad \text{if } p-q+\gamma-\beta \equiv 2 \pmod 3.$$

If $p-q \equiv 0 \pmod 3$, it has been found that the factor x^α, where $\alpha = \frac{2}{3}(p-q)-1$, can always be removed by division from the substituted equation; if a factor $x^{\alpha+\alpha'}$, where $\alpha' \geqq 1$, is to be removable in the same way, it is necessary that the exponent of x in each of the above reduced terms be as great as $\alpha+\alpha' = \frac{2}{3}(p-q)-1+\alpha'$; i. e., it is necessary that $\frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma) \geqq \frac{2}{3}(p-q)-1+\alpha'$, which requires that

$$\beta \geqq 3\alpha' - 2\gamma - 3, \quad \text{if } p-q+\gamma-\beta \equiv 0 \pmod 3.$$

Similarly must $$\beta \geqq 3\alpha' - 2\gamma - 1, \quad \text{if } p-q+\gamma-\beta \equiv 1 \pmod 3,$$

and $$\beta \geqq 3\alpha' - 2\gamma + 1, \quad \text{if } p-q+\gamma-\beta \equiv 2 \pmod 3.$$

These conditions require in general that all those terms of the general equation of (p, q) vanish, where β, or better $\beta + 2\gamma$, falls below a definite limit depending on α'. For smaller values of α', the terms which must vanish in any group

$\phi_{p-q+\gamma}$ are those of lower degrees in λ, and the larger number of terms must vanish, in general, in those groups where γ has the smaller values, i. e., where ν enters to its higher powers. The terms in the group $\phi_{p-q}\cdot\nu^q$ cannot all be made to vanish, else the species of the curve in question would be changed by a reduction in the value of q; and the highest value of α' may occur only when the single term highest in $\beta + 2\gamma$ in this group, viz., $a_{p-q,0}\lambda^{p-q}\nu^q$ alone remains; this term, under the required substitution, becomes $a_{p-q,0}x^{p-q}s^q$ and the upper limit for α' here involved is found to be $\frac{1}{3}(p-q)+1$, since $\alpha+\alpha' = \frac{2}{3}(p-q)-1+\alpha' \geqq p-q$.

Again, there must occur in the equation of the curve in question one term at least which does not involve the variable λ, else that equation would be reducible by some power of that variable. Among the possible terms in μ, ν, that involving the highest value of $\beta + 2\gamma$, and hence the highest value of α', is the term $a_{0,q}\mu^p$; this term becomes, under the required substitution, $a_{0,q}y^p$, and, on changing y^3 to x^2z as many times as possible, it is found that

$$
\begin{aligned}
& a_{0,q}y^p = a_{0,q}\,x^{\frac{2}{3}p}z^{\frac{1}{3}p}, && \text{if } p \equiv 0 \pmod 3,\\
\text{or}\quad & \phantom{a_{0,q}y^p} = a_{0,q}x^{\frac{2}{3}(p-1)}yz^{\frac{1}{3}(p-1)}, && \text{if } p \equiv 1 \pmod 3,\\
\text{or}\quad & \phantom{a_{0,q}y^p} = a_{0,q}x^{\frac{2}{3}(p-2)}y^2z^{\frac{1}{3}(p-2)}, && \text{if } p \equiv 2 \pmod 3.
\end{aligned}
$$

Hence $\alpha+\alpha'$ has for an upper limit $\frac{2}{3}p$, $\frac{2}{3}(p-1)$ or $\frac{2}{3}(p-2)$ according as $p\equiv 0$, 1 or 2 (mod 3); since $\alpha = \frac{2}{3}(p-q)-1$, this gives upper limits for α' as follows:

$$
\begin{aligned}
& \alpha' \leqq \tfrac{2}{3}q+1, && \text{if } p\equiv 0 \pmod 3,\\
\text{or}\quad & \alpha' \leqq \tfrac{2}{3}q+\tfrac{1}{3}, && \text{if } p\equiv 1 \pmod 3,\\
\text{or}\quad & \alpha' \leqq \tfrac{2}{3}q-\tfrac{1}{3}, && \text{if } p\equiv 2 \pmod 3.
\end{aligned}
$$

Then not only must α' be less than or at most equal to $\frac{1}{3}(p-q)+1$, but it must also satisfy the appropriate one of the last three conditions here found.

To restrict the general equation of (p, q) in such a way that the desired reduction by $x^{\alpha+\alpha'}$ may be made, the coefficient $a_{\beta,\gamma}$ in the assumed general equation must take the value zero whenever the conditions found above for β are not fulfilled. Examining these conditions, it appears that $a_{\beta,\gamma}$ would vanish by the third condition above, if the equations $\beta = 3\alpha' - 2\gamma$ and $p-q+\gamma-\beta \equiv 2 \pmod 3$ were both satisfied, since $\beta = 3\alpha'-2\gamma < 3\alpha'-2\gamma+1$; but if $\beta = 3\alpha'-2\gamma$, then is $p-q+\gamma-\beta = p-q+3(\gamma-\alpha') \equiv 0 \pmod 3$, the condition that $p-q+\gamma-\beta\equiv 2 \pmod 3$ is not satisfied, and $a_{\beta,\gamma}$ does not

vanish in this case. If $\beta = 3\alpha' - 2\gamma - 2$, then $p - q + \gamma - \beta = p - q + 3(\gamma - \alpha') + 2 \equiv 2 \pmod 3$, and the coefficient $a_{\beta,\gamma}$ vanishes in this case under the third condition above, since $3\alpha' - 2\gamma - 2 < 3\alpha' - 2\gamma + 1$. Also $a_{\beta,\gamma}$ must vanish for all values of β, γ where $0 \leqq \beta \leqq 3\alpha' - 2\gamma - 4$.

In order that all these conditions may be applicable to the coefficients of the equation in question, it is necessary that in the case of no one of them shall values of β, γ occur, which are not found in the coefficient $a_{\beta,\gamma}$ of some term of the equation; i. e., for all values of γ occurring, must $3\alpha' - 2\gamma - 2 \leqq p - q + \gamma$; this reduces to the condition that $\alpha' \leqq \frac{1}{3}(p - q) + \frac{2}{3} + \gamma$; difficulty arises here only when α' takes its largest value $\frac{1}{3}(p - q) + 1$, and γ has the value zero; in that case $\frac{1}{3}(p - q) + 1 > \frac{1}{3}(p - q) + \frac{2}{3} + \gamma$, and this condition fails to find application; the form of the coefficient for the application of this condition in this case is $a_{\beta,\gamma} = a_{3\alpha' - 2\gamma - 2, 0} = a_{p - q + 1, 0}$, and it is evident that the equation of the curve (p, q) contains no such coefficient. If, then, $\alpha' = \frac{1}{3}(p - q) + 1$, one condition above is inapplicable; but if $\alpha' < \frac{1}{3}(p - q) + 1$, the conditions all find application to the coefficients of the given equation.

The number of conditions imposed on the equation $\phi = 0$ in the case where $p - q \equiv 0 \pmod 3$ is then $\sum_{0}^{[\frac{3}{2}\alpha' - 1]}{}_{\gamma} (3\alpha' - 2\gamma - 2) + 1$, if $\alpha' < \frac{1}{3}(p - q) + 1$ and even, or $\sum_{0}^{[\frac{3}{2}\alpha' - 1]}{}_{\gamma} (3\alpha' - 2\gamma - 2)$, if $\alpha' < \frac{1}{3}(p - q) + 1$ and odd; this amounts to $\frac{3}{4}\alpha'(3\alpha' - 2) + 1$, if α' be even, and to $\frac{1}{4}(3\alpha' - 1)^2$, if α' be odd; hence the number of conditions imposed on the general equation of (p, q) to ensure the desired reduction by $x^{\alpha + \alpha'}$, when $\alpha' = \frac{1}{3}(p - q) + 1$, is $\frac{3}{4}\alpha'(3\alpha' - 2)$, if α' be even, and $\frac{1}{4}(3\alpha' - 1)^2 - 1$, if α' be odd and greater than unity, but unity, if $\alpha' = 1$.

The equation $\phi = 0$, after such conditions have been imposed on its coefficients, will be called the restricted equation $\phi = 0$; the curve represented by an equation thus restricted will be designated as a $(p, q)_{\alpha'}$ and the surface cutting such a curve $(p, q)_{\alpha'}$ from Σ will be denoted by S'.

The most general equation for a curve (p, q), where $p - q \equiv 0 \pmod 3$, has been found to give a curve of order $p + 2q$, which is cut from Σ by a surface S of order $\frac{1}{3}(p + 2q) + 1$, the residual intersection consisting of the cuspidal edge of Σ occurring three times, so that $f = 1$, $R = 3$, $R_i = 2$, $R_c = 1$. If the condition that $\alpha' = 1$ be imposed, the surface S' will be given by an equation of

degree $\frac{1}{3}(p+2q)$. The conditions imposed on the equation $\phi=0$ are not in general such that the restricted equation $\phi=0$ will be a reducible equation; hence, if the curve (p, q) be broken up at all to form the curve $(p, q)_{\alpha'}$, it can be brought about only by introducing into the locus of (p, q) the cuspidal edge of Σ as many times as is demanded by the value of α', that value having been reduced first by the order of the residual intersection of S and Σ; hence it will in general be true that the residual intersection of S' and Σ, if any such residual intersection exist, consists entirely of the cuspidal edge of Σ occurring as many times as the order of the residual intersection in question. This will be true generally, whether $p-q\equiv 0 \pmod 3$, or not.

In the case under consideration, where $p-q\equiv 0 \pmod 3$, the condition that $\alpha'=1$ does not cause the coefficient $a_{0,0}$ to vanish, for $\beta=\gamma=0$ makes $p-q+\gamma-\beta=p-q\equiv 0 \pmod 3$, and the first condition for β, viz., that $\beta \geqq 3\alpha'-2\gamma-3$, is satisfied here since $0=3\alpha'-3$; hence the term $a_{0,0}\mu^{p-q}\nu^q$ is not removed from the equation of (p, q); this term, under the substitutions required, becomes $a_{0,0}y^{p-q}s^q = a_{0,0}x^{\frac{2}{3}(p-q)}z^{\frac{1}{3}(p-q)}s^q$; and the rejection of the factor $x^{\alpha+\alpha'}=x^{\frac{2}{3}(p-q)}$ gives $a_{0,0}z^{\frac{1}{3}(p-q)}s^q$, a term free from both x and y; consequently the cuspidal edge of Σ does not occur at all on S' and the curve $(p, q)_{\alpha'}$ is accordingly a curve of total intersection of S' and Σ whenever $\alpha'=1$ The number of conditions imposed on the equation in this case is found by the formula given to be $\frac{1}{4}(3\alpha'-1)^2=\frac{1}{4}(3-1)^2=1$, if $\alpha'<\frac{1}{3}(p-q)+1$, i. e. if $p>q$, which is always true in the cases under consideration, the case where $p=q$ having been finally disposed of on page 277; the single condition to be applied here, as determined by the value $\beta \geqq 3\alpha'-2\gamma-2$ and $p-q+\gamma-\beta\equiv 2 \pmod 3$, is $a_{\beta,\gamma}=a_{3\alpha'-2\gamma-2,\gamma}=a_{1-2\gamma,\gamma}=a_{1,0}=0$.

In the case where α' had the value zero, the residual intersection was found to be made up of the cuspidal edge of Σ occurring three times. The rejection of $x^{\alpha'}=x$, since the plane $x=0$ contains that edge three times, simply removes the residual intersection and does not change the order of multiplicity of the vertex of Σ or of the cusp of the infinite cubic on the curve proper; i. e., π and ρ have the same values in the cases of the curve $(p, q)_1$ and the curve (p, q); viz., $\pi=p-q$ and $\rho=2q$.

The curves $(p, q)_1$, where $p-q\equiv 0 \pmod 3$, are then identical with those of Class I′ (page 279), characterized thus,

I′. $p>q,\ p-q\equiv 0 \pmod 3,\ \alpha'=1,\ a_{1,0}=0,\ m=p+2q,\ \pi=p-q,$
$\rho=2q,\ m'=\frac{1}{3}(p+2q),\ f=R=R_i=R_c=0.$

vanish in this case. If $\beta = 3\alpha' - 2\gamma - 2$, then $p - q + \gamma - \beta = p - q + 3(\gamma - \alpha') + 2 \equiv 2 \pmod 3$, and the coefficient $a_{\beta,\gamma}$ vanishes in this case under the third condition above, since $3\alpha' - 2\gamma - 2 < 3\alpha' - 2\gamma + 1$. Also $a_{\beta,\gamma}$ must vanish for all values of β, γ where $0 \leqq \beta \leqq 3\alpha' - 2\gamma - 4$.

In order that all these conditions may be applicable to the coefficients of the equation in question, it is necessary that in the case of no one of them shall values of β, γ occur, which are not found in the coefficient $a_{\beta,\gamma}$ of some term of the equation; i. e., for all values of γ occurring, must $3\alpha' - 2\gamma - 2 \leqq p - q + \gamma$; this reduces to the condition that $\alpha' \leqq \frac{1}{3}(p-q) + \frac{2}{3} + \gamma$; difficulty arises here only when α' takes its largest value $\frac{1}{3}(p-q) + 1$, and γ has the value zero; in that case $\frac{1}{3}(p-q) + 1 > \frac{1}{3}(p-q) + \frac{2}{3} + \gamma$, and this condition fails to find application; the form of the coefficient for the application of this condition in this case is $a_{\beta,\gamma} = a_{3\alpha'-2\gamma-2,\,0} = a_{p-q+1,\,0}$, and it is evident that the equation of the curve (p, q) contains no such coefficient. If, then, $\alpha' = \frac{1}{3}(p-q) + 1$, one condition above is inapplicable; but if $\alpha' < \frac{1}{3}(p-q) + 1$, the conditions all find application to the coefficients of the given equation.

The number of conditions imposed on the equation $\phi = 0$ in the case where $p - q \equiv 0 \pmod 3$ is then $\sum_{0}^{[\frac{3}{2}\alpha'-1]} {}_{\gamma} (3\alpha' - 2\gamma - 2) + 1$, if $\alpha' < \frac{1}{3}(p-q) + 1$ and even, or $\sum_{0}^{[\frac{3}{2}\alpha'-1]} {}_{\gamma} (3\alpha' - 2\gamma - 2)$, if $\alpha' < \frac{1}{3}(p-q) + 1$ and odd; this amounts to $\frac{3}{4}\alpha'(3\alpha' - 2) + 1$, if α' be even, and to $\frac{1}{4}(3\alpha' - 1)^2$, if α' be odd; hence the number of conditions imposed on the general equation of (p, q) to ensure the desired reduction by $x^{\alpha+\alpha'}$, when $\alpha' = \frac{1}{3}(p-q) + 1$, is $\frac{3}{4}\alpha'(3\alpha' - 2)$, if α' be even, and $\frac{1}{4}(3\alpha' - 1)^2 - 1$, if α' be odd and greater than unity, but unity, if $\alpha' = 1$.

The equation $\phi = 0$, after such conditions have been imposed on its coefficients, will be called the restricted equation $\phi = 0$; the curve represented by an equation thus restricted will be designated as a $(p, q)_{\alpha'}$ and the surface cutting such a curve $(p, q)_{\alpha'}$ from Σ will be denoted by S'.

The most general equation for a curve (p, q), where $p - q \equiv 0 \pmod 3$, has been found to give a curve of order $p + 2q$, which is cut from Σ by a surface S of order $\frac{1}{3}(p + 2q) + 1$, the residual intersection consisting of the cuspidal edge of Σ occurring three times, so that $f = 1$, $R = 3$, $R_i = 2$, $R_c = 1$. If the condition that $\alpha' = 1$ be imposed, the surface S' will be given by an equation of

degree $\frac{1}{3}(p+2q)$. The conditions imposed on the equation $\phi=0$ are not in general such that the restricted equation $\phi=0$ will be a reducible equation; hence, if the curve (p, q) be broken up at all to form the curve $(p, q)_{\alpha'}$, it can be brought about only by introducing into the locus of (p, q) the cuspidal edge of Σ as many times as is demanded by the value of α', that value having been reduced first by the order of the residual intersection of S and Σ; hence it will in general be true that the residual intersection of S' and Σ, if any such residual intersection exist, consists entirely of the cuspidal edge of Σ occurring as many times as the order of the residual intersection in question. This will be true generally, whether $p-q\equiv 0 \pmod 3$, or not.

In the case under consideration, where $p-q\equiv 0 \pmod 3$, the condition that $\alpha'=1$ does not cause the coefficient $a_{0,0}$ to vanish, for $\beta=\gamma=0$ makes $p-q+\gamma-\beta=p-q\equiv 0 \pmod 3$, and the first condition for β, viz., that $\beta \geqq 3\alpha'-2\gamma-3$, is satisfied here since $0=3\alpha'-3$; hence the term $a_{0,0}\mu^{p-q}\nu^q$ is not removed from the equation of (p, q); this term, under the substitutions required, becomes $a_{0,0}y^{p-q}s^q=a_{0,0}x^{\frac{2}{3}(p-q)}z^{\frac{1}{3}(p-q)}s^q$; and the rejection of the factor $x^{\alpha+\alpha'}=x^{\frac{2}{3}(p-q)}$ gives $a_{0,0}z^{\frac{1}{3}(p-q)}s^q$, a term free from both x and y; consequently the cuspidal edge of Σ does not occur at all on S' and the curve $(p, q)_{\alpha'}$ is accordingly a curve of total intersection of S' and Σ whenever $\alpha'=1$ The number of conditions imposed on the equation in this case is found by the formula given to be $\frac{1}{4}(3\alpha'-1)^2=\frac{1}{4}(3-1)^2=1$, if $\alpha'<\frac{1}{3}(p-q)+1$, i. e. if $p>q$, which is always true in the cases under consideration, the case where $p=q$ having been finally disposed of on page 277; the single condition to be applied here, as determined by the value $\beta \geqq 3\alpha'-2\gamma-2$ and $p-q+\gamma-\beta\equiv 2 \pmod 3$, is $a_{\beta,\gamma}=a_{3\alpha'-2\gamma-2,\gamma}=a_{1-2\gamma,\gamma}=a_{1,0}=0$.

In the case where α' had the value zero, the residual intersection was found to be made up of the cuspidal edge of Σ occurring three times. The rejection of $x^{\alpha'}=x$, since the plane $x=0$ contains that edge three times, simply removes the residual intersection and does not change the order of multiplicity of the vertex of Σ or of the cusp of the infinite cubic on the curve proper; i. e., π and ρ have the same values in the cases of the curve $(p, q)_1$ and the curve (p, q); viz., $\pi=p-q$ and $\rho=2q$.

The curves $(p, q)_1$, where $p-q\equiv 0 \pmod 3$, are then identical with those of Class I′ (page 279), characterized thus,

I′. $p>q, p-q\equiv 0 \pmod 3, \alpha'=1, a_{1,0}=0, m=p+2q, \pi=p-q,$
$$\rho=2q, m'=\tfrac{1}{3}(p+2q), f=R=R_i=R_c=0.$$

If $\alpha' \geqq 2$, the natures of the curve $(p, q)_{\alpha'}$ resulting, and of the surface S' cutting the same from Σ, are found as follows. The order of the surface S is $\frac{1}{3}(p+2q)+1$; hence the order of the surface S' must be less by α' than that number, i. e., $\frac{1}{3}(p+2q)+1-\alpha'$. It has been seen that the coefficient $a_{\beta,\gamma}$, where $\beta = 3\alpha' - 2\gamma - 3$, is not made to vanish for any value of α'; since, in any group $\phi_{p-q+\gamma}$, the variable λ occurs in general in some term to the power $p-q+\gamma$, there will occur in the equation of the curve $(p, q)_{\alpha'}$ a term where $\beta = 3\alpha' - 2\gamma - 3$ if $0 \leqq 3\alpha' - 2\gamma - 3 \leqq p - q + \gamma$; the former condition is satisfied when $\gamma = 0$ and $\alpha' \geqq 2$; the latter condition demands that $3\alpha' \leqq p - q + 3\gamma + 3$ and is always fulfilled, since $\alpha' \leqq \frac{1}{3}(p-q)+1$; hence the equation of $(p, q)_{\alpha'}$ always contains a term of the form

$$a_{\beta,\gamma}\lambda^{\beta}\mu^{p-q+\gamma-\beta}\nu^{q-\gamma} = a_{3\alpha'-3\gamma-3,\gamma}\lambda^{3\alpha'-2\gamma-3}\mu^{p-q+3\gamma-3\alpha'+3}\nu^{q-\gamma},$$

which becomes, under the required substitution,

$$a_{3\alpha'-2\gamma-3,\gamma}\,x^{3\alpha'-2\gamma-3}y^{p-q+3\gamma-3\alpha'+3}s^{q-\gamma}.$$

If in this term x^2z be substituted for y^3 as many times as is possible, it takes the form $a_{3\alpha'-2\gamma-3,\gamma}\,x^{\frac{2}{3}(p-q)-1+\alpha'}z^{\frac{1}{3}(p-q)+\gamma-\alpha'+1}s^{q-\gamma}$. Rejecting the factor $x^{\alpha+\alpha'} = x^{\frac{2}{3}(p-q)-1+\alpha'}$ from this term leaves only $a_{3\alpha'-2\gamma-3,\gamma}\,z^{\frac{1}{3}(p-q)+\gamma-\alpha'+1}s^{q-\gamma}$, which is of the zeroth degree in x, y; consequently, the equation of the surface S', which cuts the curve $(p, q)_{\alpha'}$ from Σ, where $p-q \equiv 0 \pmod 3$, is of the zeroth degree in the variables x, y whenever $\alpha' \geqq 2$. The residual intersection, consisting entirely in every case of the cuspidal edge of Σ occurring a certain number of times, is wanting, since the surface S' does not contain that edge at all. But the rejection of the factor $x^{\alpha'}$ from the equation of the surface S has reduced the common intersection of the two surfaces by the cuspidal edge counting $3\alpha'$ times, and has consequently not only removed the former residual intersection of order three, but has also subjected the curve to a reduction of its order by $3(\alpha'-1)$, since the cuspidal edge has been rejected that number of times from the curve portion of the intersection. Hence, in this case, where $p-q \equiv 0 \pmod 3$, the restricted equation $\phi = 0$ represents a curve $(p, q)_{\alpha'}$ of order $m = p + 2q - 3(\alpha'-1)$; and this curve $(p, q)_{\alpha'}$ is a curve of total intersection of the newly found surface S' with Σ.

The rejection of the cuspidal edge of Σ thus $3(\alpha'-1)$ times from the curve (p, q) effects a reduction by that number in the order of multiplicity of the vertex of Σ as a point of the curve; in the case of the curve (p, q) it has been

found that $\pi = p - q$; hence, for the curve $(p, q)_{\alpha'}$, $\pi = p - q - 3(\alpha' - 1)$. Similarly, the curve $(p, q)_{\alpha'}$ passes $3(\alpha' - 1)$ less times through the cusp of the infinite cubic than does the curve (p, q); consequently, $\rho = 2q - 3(\alpha' - 1)$. Since neither π nor ρ can be negative, the upper limits found already for α' are here made evident again; and here, as in all other cases of curves of total intersection thus far considered, $\pi \equiv 0 \pmod 3$. If both π and ρ have in any case the value zero, then must the curve in question be of a more general character than the (p, q) of the same order, which is known to pass $p - q$ times through the vertex of Σ and $2q$ times through the cusp of the infinite cubic (since both $p - q$ and $2q$ can in no case have the value zero). The curve $(p, q)_{\alpha'}$, which is, from this point of view, one of the most general curves on Σ, occurs when $\pi = p - q - 3(\alpha' - 1) = \rho = 2q - 3(\alpha' - 1) = 0$, which demands that $p = 3q$ and $\alpha' = \frac{2}{3}q + 1 = \frac{2}{9}p + 1$; while $m = p$, $m' = \frac{1}{3}p$, and $2q = p - q \equiv 0 \pmod 3$, making $q \equiv 0 \pmod 3$ and $p \equiv 0 \pmod 9$. Such a curve is the $(9,3)_3$, where $\alpha = 3$ and $\alpha' = 3$, while $m = 9$, $m' = 3$, and $\pi = \rho = 0$; this case may now be worked out according to the conditions already given, as follows: The most general equation of the curve (9,3) is of the form

$$\begin{aligned} &a_{9,3}\lambda^9 + a_{8,3}\lambda^8\mu + a_{7,3}\lambda^7\mu^2 + a_{6,3}\lambda^6\mu^3 + a_{5,3}\lambda^5\mu^4 + a_{4,3}\lambda^4\mu^5 + a_{3,3}\lambda^3\mu^6 + a_{2,3}\lambda^2\mu^7 \\ &\quad + a_{1,3}\lambda\mu^8 + a_{0,3}\mu^9 + (a_{8,2}\lambda^8 + a_{7,2}\lambda^7\mu + a_{6,2}\lambda^6\mu^2 + a_{5,2}\lambda^5\mu^3 + a_{4,2}\lambda^4\mu^4 \\ &\quad + a_{3,2}\lambda^3\mu^5 + a_{2,2}\lambda^2\mu^6 + a_{1,2}\lambda\mu^7 + a_{0,2}\mu^8)\nu + (a_{7,1}\lambda^7 + a_{6,1}\lambda^6\mu + a_{5,1}\lambda^5\mu^2 \\ &\quad + a_{4,1}\lambda^4\mu^3 + a_{3,1}\lambda^3\mu^4 + a_{2,1}\lambda^2\mu^5 + a_{1,1}\lambda\mu^6 + a_{0,1}\mu^7)\nu^2 + (a_{6,0}\lambda^6 + a_{5,0}\lambda^5\mu \\ &\quad + a_{4,0}\lambda^4\mu^2 + a_{3,0}\lambda^3\mu^3 + a_{2,0}\lambda^2\mu^4 + a_{1,0}\lambda\mu^5 + a_{0,0}\mu^6)\nu^3 = 0. \end{aligned}$$

Imposing the condition that all terms vanish where $\beta = 3\alpha' - 2\gamma - 2 = 7 - 2\gamma$, and where $\beta \leqq 3\alpha' - 2\gamma - 4 = 5 - 2\gamma$, removes $\frac{1}{4}(3\alpha' - 1)^2 - 1 = 15$ terms, since it makes $a_{1,3} = a_{3,2} = a_{1,2} = a_{0,2} = a_{5,1} = a_{3,1} = a_{2,1} = a_{1,1} = a_{0,1} = a_{5,} = a_{4,0} = a_{3,0} = a_{2,0} = a_{1,0} = a_{0,0} = 0$; this leaves the equation of the curve $(p, q)_{\alpha'}$ in the form

$$\begin{aligned} &a_{9,3}\lambda^9 + a_{8,3}\lambda^8\mu + a_{7,3}\lambda^7\mu^2 + a_{6,3}\lambda^6\mu^3 + a_{5,3}\lambda^5\mu^4 + a_{4,3}\lambda^4\mu^5 + a_{3,3}\lambda^3\mu^6 \\ &\quad + a_{2,3}\lambda^2\mu^7 + a_{0,3}\mu^9 + (a_{8,2}\lambda^8 + a_{7,2}\lambda^7\mu + a_{6,2}\lambda^6\mu^2 + a_{5,2}\lambda^5\mu^3 + a_{4,2}\lambda^4\mu^4 \\ &\quad + a_{2,2}\lambda^2\mu^5)\nu + (a_{7,1}\lambda^7 + a_{6,1}\lambda^6\mu + a_{4,1}\lambda^4\mu^3)\nu^2 + a_{6,0}\lambda^6\nu^3 = 0. \end{aligned}$$

Substituting x, y and s for λ, μ and ν respectively, changing y^3 to x^2z wherever possible, and removing the factor $x^{\alpha+\alpha'}=x^6$, there remains

$$\begin{aligned}a_{9,3}x^3 &+ a_{8,3}x^2y + a_{7,3}xy^2 + a_{6,3}x^2z + a_{5,3}xyz + a_{4,3}y^2z + a_{3,3}xz^2 + a_{2,3}yz^2\\ &+ a_{0,3}z^3 + a_{8,2}x^2s + a_{7,2}xys + a_{6,2}y^2s + a_{5,2}xzs + a_{4,2}yzs + a_{2,2}z^2s\\ &+ a_{7,1}xs^2 + a_{6,1}ys^2 + a_{4,1}zs^2 + a_{6,0}s^3 = 0,\end{aligned}$$

the equation of the cubic surface S', whose intersection with Σ is the curve $(9, 3)_3$ of order $m = p + 2q - 3(\alpha' - 1) = 9$. This equation contains all the terms of the most general homogeneous cubic equation in the four variables x, y, z, s except the term in y^3. Suppose there had occurred the terms $a'_{6,3}y^3$ and $a''_{6,3}x^2z$ in this equation, making it represent the general cubic surface; the substitution of x^2z for y^3 in the former term would not have affected the intersection of the surface with Σ, and the two terms $a'_{6,3}y^3$ and $a''_{6,3}x^2z$, now $a'_{6,3}x^2z$ and $a''_{6,3}x^2z$, could have been combined as a single term $a_{6,3}x^2z$; hence, the intersection of this surface S' with Σ gives the most general curve of total intersection that can be cut from Σ by a cubic surface; such intersection is a general curve of order 9, and is deservedly ranked as a curve of more general character than those which are required to pass a certain number of times through the vertex of Σ or the cusp of the infinite cubic, or both. Similarly, the general sextic surface cuts from Σ a curve of the species $(18, 6)_5$ and of order 18, a curve of total intersection, where $\alpha = 7$ and $\alpha' = 5$. And in the same manner may be found here the curves of total intersection of Σ with the general surfaces S of orders 9, 12, 15, 18, 21, etc., those curves being of the species $(27, 9)_7$, $(36, 12)_9$' $(45, 15)_{11}$, etc.

The curves of total intersection here occurring will be grouped as Class II′, and may be characterized thus:

II′. $p > q$, $p - q \equiv 0 \pmod 3$, $\alpha' \geqq 1$, $m = p + 2q - 3(\alpha' - 1)$, $\pi = p - q - 3(\alpha' - 1)$, $\rho = 2q - 3(\alpha' - 1)$, $m' = \frac{1}{3}(p + 2q) - \alpha' + 1$, $f = R = R_i = R_c = 0$.

Class II′ as here described, evidently includes the curves of Class I′, as those curves were designated on page 279; hence, the latter class will no more be mentioned as a distinct group, but will be regarded as coming in with Class II′

This completes for the present the discussion of the curves $(p, q)_{\alpha'}$, where $p - q \equiv 0 \pmod 3$.

If $p - q \equiv 1 \pmod 3$, it is known that the factor x^{α}, where α has the value $\frac{2}{3}(p - q - 1)$, can be rejected in every case from the equation obtained from $\phi = 0$ by the required substitutions. The conditions under which a factor $x^{\alpha + \alpha'}$, where $\alpha' \geqq 1$, can be divided out from that equation are readily found by the method employed in the preceding case. For that purpose the exponent of x in each term of the equation in question must be as great as $\alpha + \alpha' = \frac{2}{3}(p - q - 1) + \alpha'$; i. e., it is necessary that $\frac{1}{3}\beta + \frac{2}{3}(p - q + \gamma)$, as found on page 282, be as great as $\frac{2}{3}(p - q - 1) + \alpha'$, which requires that

$$\beta \geqq 3\alpha' - 2\gamma - 2, \text{ if } p - q + \gamma - \beta \equiv 0 \pmod 3.$$

Likewise must

$$\beta \geqq 3\alpha' - 2\gamma, \quad \text{if } p - q + \gamma - \beta \equiv 1 \pmod 3,$$

and

$$\beta \geqq 3\alpha' - 2\gamma + 2, \text{ if } p - q + \gamma - \beta \equiv 2 \pmod 3.$$

These conditions are entirely similar in form and application to those already discussed for the case where $p - q = 0 \pmod 3$. Hence, since the terms in the group ϕ_{p-q} cannot all be wanting, the largest value is allowed to α' when the only non-vanishing coefficient in that group is $a_{p-q,0}$; this leaves a term $a_{p-q,0}\lambda^{p-q}\nu^q$ which becomes $a_{p-q,0}x^{p-q}s^q$; hence $\alpha + \alpha' \leqq p - q$, which gives for α' an upper limit, $\alpha' \leqq \frac{1}{3}(p - q + 2)$. There must also occur in the equation of $(p, q)_{\alpha'}$ a term free from λ; and the one giving the highest limit for $\beta + 2\gamma$, and, hence, for α', is $a_{0,q}\mu^p$; this term becomes $a_{0,q}y^p$ and

$$a_{0,q}y^p = a_{0,q}x^{\frac{2}{3}p}z^{\frac{1}{3}p}, \qquad \text{if } p \equiv 0 \pmod 3,$$

or

$$= a_{0,q}x^{\frac{2}{3}(p-1)}yz^{\frac{1}{3}(p-1)}, \text{ if } p \equiv 1 \pmod 3,$$

or

$$= a_{0,q}x^{\frac{2}{3}(p-2)}y^2z^{\frac{1}{3}(p-2)}, \text{ if } p \equiv 2 \pmod 3,$$

as given on page 283. Consequently, $\alpha + \alpha'$ has an upper limit $\frac{2}{3}p$, $\frac{2}{3}(p - 1)$ or $\frac{2}{3}(p - 2)$ according as $p \equiv 0$, 1 or 2 (mod 3); since $\alpha = \frac{2}{3}(p - q - 1)$, this gives as upper limits for α' that

$$\alpha' \leqq \tfrac{2}{3}q + \tfrac{2}{3}, \text{ if } p \equiv 0 \pmod 3,$$

or

$$\leqq \tfrac{2}{3}q, \qquad \text{if } p \equiv 1 \pmod 3,$$

or

$$\leqq \tfrac{2}{3}q - \tfrac{2}{3}, \text{ if } p \equiv 2 \pmod 3.$$

Not only must the appropriate one of these last three conditions be obeyed by α', but also in every case must $\alpha' \leqq \frac{1}{3}(p-q+2)$.

To obtain the equation of the curve $(p, q)_{\alpha'}$ from that of (p, q), all terms of the latter, which have β falling below the limits found above, must vanish. If $\beta = 3\alpha' - 2\gamma + 1$, $p - q + \gamma - \beta = p - q - 3\alpha' + 3\gamma - 1 \equiv p - q - 1 \equiv 0 \pmod 3$, and the coefficient $a_{3\alpha' - 2\gamma + 1, \gamma}$ is not required to vanish. If $\beta = 3\alpha' - 2\gamma - 1$, $p - q + \gamma - \beta = p - q - 3\alpha' + 3\gamma + 1 \equiv p - q + 1 \equiv 2 \pmod 3$, and the coefficient $a_{3\alpha' - 2\gamma - 1, \gamma}$ must be made to vanish, under the requirements of the third condition for β above, viz., that $\beta \geqq 3\alpha' - 2\gamma + 2$ if $p - q + \gamma - \beta \equiv 2 \pmod 3$. These conditions require not only that $a_{\beta, \gamma} = 0$, when $\beta = 3\alpha' - 2\gamma - 1$, but also whenever $0 \leq \beta \leq 3\alpha' - 2\gamma - 3$. But if in any case these conditions require an $a_{\beta, \gamma}$ to vanish for a combination of values of β and γ which do not occur together in any coefficient of the equation $\phi = 0$, such a condition must not be counted in making up the number of restrictions to be actually imposed on the coefficients of the equation in question. The highest value of β reached in the conditions is $\beta = 3\alpha' - 2\gamma - 1$; that will surpass the highest power of λ appearing in the equation $\phi = 0$ when and only when $3\alpha' - 2\gamma - 1 > p - q + \gamma$; if α' takes its largest value $\frac{1}{3}(p-q+2)$, the condition becomes $1 > 3\gamma$, which is unsatisfied except when $\gamma = 0$; hence, none of the conditions fails to be applicable when $\alpha' < \frac{1}{3}(p-q+2)$, and only a single one of them is inapplicable when $\alpha' = \frac{1}{3}(p-q+2)$; the coefficient, the vanishing of which the inapplicable condition demands, is $a_{3\alpha'-2\gamma-1, \gamma} = a_{p-q+1, 0}$, and it is evident that no such coefficient ever appears in the equation of the curve (p, q).

The number of conditions imposed on the coefficients of the equation of the curve (p, q) to make it become the equation of the curve $(p, q)_{\alpha'}$, in the case where $p - q \equiv 1 \pmod 3$, is then $\sum_{\gamma=0}^{[\frac{3}{2}\alpha' - \frac{1}{2}]} (3\alpha' - 2\gamma - 1)$, if $\alpha' < \frac{1}{3}(p-q+2)$, and that number diminished by unity if $\alpha' = \frac{1}{3}(p-q+2)$; this sum amounts to $\frac{1}{4}(3\alpha')^2$, if α' is even, and to $\frac{1}{4}(3\alpha' - 1)(3\alpha' + 1)$, if α' is odd; hence the number of conditions imposed, when $\alpha' < \frac{1}{3}(p-q+2)$, is $\frac{1}{4}(3\alpha')^2$ or $\frac{1}{4}(9\alpha'^2 - 1)$, according as α' is even or odd; while, if $\alpha' = \frac{1}{3}(p-q+2)$, the corresponding number is $\frac{1}{4}(3\alpha')^2 - 1$ or $\frac{1}{4}(9\alpha'^2 - 1) - 1$, according as α' is even or odd; but the number is always two, if $\alpha' = 1$.

The general equation $\phi = 0$, where $p - q \equiv 1 \pmod 3$, has been found to represent a curve (p, q) of order $p + 2q$ which is cut from Σ by a surface S of order $\frac{1}{3}(p + 2q + 2)$, where the residual intersection consists of the cuspidal edge of Σ occurring twice, so that $f = 1$, $R = 2$, $R_i = 2$, and $R_c = 0$. If now the condition that $\alpha' \geqq 1$ be imposed, the surface S' cutting the curve $(p, q)_{\alpha'}$ from Σ is clearly of order $\frac{1}{3}(p + 2q + 2) - \alpha'$. The conditions imposed to make the equation of S reducible by $x^{\alpha'}$ will, if $\alpha' \geqq 1$, require that $(p, q)_{\alpha'}$ be a curve of lower order than the corresponding (p, q), since the total intersection of S and Σ suffers a reduction by the cuspidal edge of Σ, occurring $3\alpha' \geqq 3$ times, while the residual intersection for the curve (p, q) consists of that edge occurring only twice. But it is evident that the curve (p, q) is not in general subject to any other reduction, and the curve $(p, q)_{\alpha'}$ will consequently be in general a proper curve, and any residual intersection of S' and Σ will be made up entirely of the cuspidal edge of Σ. The coefficient $a_{\beta, \gamma}$, where $\beta = 3\alpha' - 2\gamma - 2$ and $p - q + \gamma - \beta \equiv 0 \pmod 3$, does not vanish under the conditions of restriction here imposed; this condition is possible of satisfaction for β in every case, since $3\alpha' - 2\gamma - 2 \geqq 0$, if $\gamma = 0$ and $\alpha' \geqq 1$; hence there occurs in the equation of the surface S' the term derived from that one in $\phi = 0$ of the form

$$a_{3\alpha' - 2\gamma - 2, \gamma} \lambda^{3\alpha' - 2\gamma - 2} \mu^{p - q + \gamma - \beta} \nu^{q - \gamma};$$

this term, on performing the required substitutions and reductions, becomes

$$a_{3\alpha' - 2\gamma - 2, \gamma} x^{3\alpha' - 2\gamma - 2 + \frac{2}{3}(p - q) + 2\gamma - 2\alpha' + \frac{4}{3}} z^{\frac{1}{3}(p - q) + \gamma - \alpha + \frac{2}{3}} s^{q - \gamma},$$

which is again

$$a_{3\alpha' - 2\gamma - 2, \gamma} x^{\frac{2}{3}(p - q - 1) + \alpha'} z^{\frac{1}{3}(p - q + 2) + \gamma - \alpha'} s^{q - \gamma}.$$

On rejecting the factor $x^{\alpha + \alpha'} = x^{\frac{2}{3}(p - q - 1) + \alpha'}$, this term becomes

$$a_{3\alpha' - 2\gamma - 2, \gamma} z^{\frac{1}{3}(p - q + 2) + \gamma - \alpha'} s^{q - \gamma},$$

which is of the zeroth degree in x, y; consequently the surface S' does not contain the cuspidal edge of Σ at all, and the curve $(p, q)_{\alpha'}$ is accordingly a curve of total intersection of Σ and S'. In the case of the curve (p, q), it has been found that $\pi = p - q$ and $\rho = 2q$; since the cuspidal edge of Σ has been rejected $3\alpha'$ times, and since the former residual intersection consisted of that

edge occurring twice, it follows that in the case of the curve $(p, q)_{a'}$, where $p - q \equiv 1 \pmod 3$, $\pi = p - q - 3a' + 2$ and $\rho = 2q - 3a' + 2$. The upper limits here involved for a' agree with those already found for the case in question. Here, again, the curves $(p, q)_{a'}$ of total intersection have $\pi \equiv 0 \pmod 3$. If both π and ρ have the value zero, then must the curve represented pass through neither the vertex of Σ nor the cusp of the infinite cubic, and hence the curve will be of more general character than the (p, q) of the same order, which has been found to pass a certain number of times through one or both of those points. Such a curve $(p, q)_{a'}$ occurs when $\pi = p - q - 3a' + 2 = \rho = 2q - 3a' + 2 = 0$, which demands that $p = 3q$ and $a' = \frac{2}{3}q + \frac{2}{3} = \frac{2}{9}p + \frac{2}{3}$, while $m = p$, $m' = \frac{1}{3}p$ and $2q = p - q \equiv 1 \pmod 3$, making $q \equiv 2 \pmod 3$; the $(6,2)_2$ answering these requirements,—that $a' = 2$, etc.,—is cut from Σ by the surface S' whose equation is determined as follows:

The general equation of the curve (6,2) is of the form:

$$a_{6,2}\lambda^6 + a_{5,2}\lambda^5\mu + a_{4,2}\lambda^4\mu^2 + a_{3,2}\lambda^3\mu^3 + a_{2,2}\lambda^2\mu^4 + a_{1,2}\lambda\mu^5 + a_{0,2}\mu^6 + (a_{5,1}\lambda^5 + a_{4,1}\lambda^4\mu + a_{3,1}\lambda^3\mu^2 + a_{2,1}\lambda^2\mu^3 + a_{1,1}\lambda\mu^4 + a_{0,1}\mu^5)\nu + (a_{4,0}\lambda^4 + a_{3,0}\lambda^3\mu + a_{2,0}\lambda^2\mu^2 + a_{1,0}\lambda\mu^3 + a_{0,0}\mu^4)\nu^2 = 0.$$

The conditions to be imposed on the coefficients, in order that a' may have the value two, are found by the method used above to demand that $a_{1,2} = a_3 = a_{1,1} = a_{0,1} = a_{3,0} = a_{2,0} = a_{1,0} = a_{0,0} = 0$; this leaves the equation of the curve $(6,2)_2$ which is of the form:

$$a_{6,2}\lambda^6 + a_{5,2}\lambda^5\mu + a_{4,2}\lambda^4\mu^2 + a_{3,2}\lambda^3\mu^3 + a_{2,2}\lambda^2\mu^4 + a_{0,2}\mu^6 + a_{5,1}\lambda^5\nu + a_{4,1}\lambda^4\mu\nu + a_{2,1}\lambda^2\mu^3\nu + a_{4,0}\lambda^4\nu^2 = 0.$$

Performing the required substitution, this becomes

$$a_{6,2}x^6 + a_{5,2}x^5y + a_{4,2}x^4y^2 + a_{3,2}x^3y^3 + a_{2,2}x^2y^4 + a_{0,2}y^6 + a_{5,1}x^5s + a_{4,1}x^4ys + a_{2,1}x^2y^2s + a_{4,0}x^4s^2 = 0;$$

changing y^3 to x^2z as many times as possible in each term of this equation, and rejecting the factor $x^{a+a'} = x^4$, there results the equation of the surface S' of the form:

$$a_{6,2}x^2 + a_{5,2}xy + a_{4,2}y^2 + a_{3,2}xz + a_{2,2}yz + a_{0,2}z^2 + a_{5,1}xs + a_{4,1}ys + a_{2,1}zs + a_{4,0}s^2 = 0.$$

This equation is that of the general quadric surface, and thus it is evident that the curve $(6,2)_2$ is the most general curve which can be cut from Σ by a quadric surface.

Similarly, a general curve of order 15 is the $(15,5)_4$ cut from Σ by a general surface of order 5; likewise, the $(24, 8)_6$, a curve of order 24, is the total intersection of the cone Σ with a general surface of order 8, the $(33, 11)_8$ of order 33 is the total intersection of Σ with a general surface of order 11, the $(42, 14)_{10}$, of order 42, is the total intersection of Σ with a general surface of order 14, etc., etc.

These special curves of total intersection $(p, q)_{\alpha'}$ will comprise Class III′ and may be characterized thus:

III′. $p-q\equiv 1 \pmod 3$, $\alpha' \geqq 1$, $m=p+2q-3\alpha'+2$, $\pi=p-q-3\alpha'+2$,
$\rho=2q-3\alpha'+2$, $m'=\frac{1}{3}(p+2q+2)-\alpha'$, $f=R=R_i=R_c=0$.

If $p-q\equiv 2 \pmod 3$, it has been proved that x^{α}, where $\alpha=\frac{2}{3}(p-q-2)$, is a factor of all the terms of the equation obtained from $\phi=0$ by the required substitutions. The conditions under which $x^{\alpha+\alpha'}$, where $\alpha'\geqq 1$, becomes a factor in the same equation, are readily found by the method used in the two cases already discussed. For the occurrence of such a factor, it is necessary that the power of x in each term be as great as $\alpha+\alpha'=\frac{2}{3}(p-q-2)+\alpha'$; i. e., it is necessary that $\frac{1}{3}\beta+\frac{2}{3}(p-q+\gamma)$, as given on page 282, be as great as $\frac{2}{3}(p-q-2)+\alpha'$, which requires that

$$\beta \geqq 3\alpha'-2\gamma-4, \text{ if } p-q+\gamma-\beta\equiv 0 \pmod 3.$$

Likewise must

$$\beta \geqq 3\alpha'-2\gamma-2, \text{ if } p-q+\gamma-\beta\equiv 1 \pmod 3,$$

or

$$\beta \geqq 3\alpha'-2\gamma, \quad \text{ if } p-q+\gamma-\beta\equiv 2 \pmod 3.$$

As in the preceding cases, the terms in the group ϕ_{p-q} cannot all be wanting, and the highest value of α' is allowed when the only coefficient remaining in that group is $a_{p-q,0}$; this leaves the term $a_{p-q,0}\lambda^{p-q}\nu^q$ present, which term becomes $a_{p-q,0}x^{p-q}s^q$; hence $\alpha+\alpha'\leqq p-q$, whence $\alpha'\leqq p-q-\alpha$, giving as an upper limit for α' that $\alpha'\leqq\frac{1}{3}(p-q+4)$. The restricted equation $\phi=0$ must contain a term free from λ and hence of the form $a_{0,\gamma}\mu^{p-q+\gamma}\nu^{q-\gamma}$; this term will permit

the highest value of $\beta + 2\gamma$, and thus of α', when $\gamma = q$; then it is of the form $a_{0,q}\mu^p$, which becomes $a_{0,q}y^p$; this reduces to

$$a_{0,q}\, x^{\frac{2}{3}p} z^{\frac{1}{3}p}, \qquad \text{if } p \equiv 0 \pmod 3,$$

or

$$a_{0,q}\, x^{\frac{2}{3}(p-1)} y z^{\frac{1}{3}(p-1)}, \quad \text{if } p \equiv 1 \pmod 3,$$

or

$$a_{0,q}\, x^{\frac{2}{3}(p-2)} y^2 z^{\frac{1}{3}(p-2)}, \quad \text{if } p \equiv 2 \pmod 3,$$

as seen on page 283. Consequently, $\alpha + \alpha'$ has as an upper limit $\frac{2}{3}p$, $\frac{2}{3}(p-1)$ or $\frac{2}{3}(p-2)$, according as $p \equiv 0, 1$ or $2 \pmod 3$; since $\alpha = \frac{2}{3}(p-q-2)$, this gives as upper limits for α' the requirement that $\alpha' \leqq \frac{2}{3}q + \frac{4}{3}$, $\frac{2}{3}q + \frac{2}{3}$ or $\frac{2}{3}q$ according as $p \equiv 0, 1$ or $2 \pmod 3$. Not only must the appropriate one of these last three conditions be satisfied by α', but also, in every case, the condition found above,—that $\alpha' \leqq \frac{1}{3}(p-q+4)$,—must be fulfilled.

The equation of (p, q) is made to become the equation of $(p, q)_{\alpha'}$ by rejecting from it all those terms in which β falls below the limits found above for it. If $\beta = 3\alpha' - 2\gamma - 1$, then $p - q + \gamma - \beta = p - q + \gamma - (3\alpha' - 2\gamma - 1) = p - q + 3(\gamma - \alpha') + 1 \equiv 1 \pmod 3$, and no coefficient $a_{\beta,\gamma}$ need vanish for such a value of β. If $\beta = 3\alpha' - 2\gamma - 3$, then $p - q + \gamma - \beta = p - q + \gamma - (3\alpha' - 2\gamma - 3) = p - q + 3(\gamma - \alpha' + 1) \equiv 2 \pmod 3$, and the coefficient $a_{3\alpha'-2\gamma-3,\gamma}$ must vanish. Likewise must every coefficient $a_{\beta,\gamma}$ vanish where $0 \leqq \beta \leqq 3\alpha' - 2\gamma - 5$. It may happen here that β, γ and α' may satisfy the conditions for such an arrangement of values of β and γ as does not occur in the equation $\phi = 0$, in which case the condition in question should not be counted in making up the total number of conditions to be imposed on that equation. Such an inapplicable condition occurs when $\beta = 3\alpha' - 2\gamma - 3$ is greater than the power of λ occurring with the value of γ in question, i. e., when $3\alpha' - 2\gamma - 3 > p - q + \gamma$; since $\alpha' \leqq \frac{1}{3}(p - q + 4)$, such a case can arise only when $p - q + 4 - 2\gamma - 3 > p - q + \gamma$, i. e., when $1 > 3\gamma$, and this is satisfied only when $\gamma = 0$; hence, one of the above conditions is inapplicable when and only when $\alpha' = \frac{1}{3}(p - q + 4)$ and $\gamma = 0$. The coefficient required to vanish in this case is $a_{p-q+1,0}$, and no such coefficient appears in the equation $\phi = 0$.

The number of conditions imposed on the coefficients of the equation $\phi = 0$ to obtain the equation of the curve $(p, q)_{\alpha'}$ in the case in question, where $p - q \equiv 2 \pmod 3$, is $\sum_{0}^{[\frac{3}{2}(\alpha'-1)]}\gamma\,(3\alpha' - 2\gamma - 3)$, if $\alpha' < \frac{1}{3}(p - q + 4)$; this sum amounts to

$\frac{1}{4}(3\alpha' - 2)^2$, if α' is even, and to $\frac{3}{4}(\alpha' - 1)(3\alpha' - 1)$ if α' be odd and greater than unity, but to unity, if $\alpha' = 1$; hence, the number of conditions imposed, when $\alpha' = \frac{1}{3}(p - q + 4)$, is $\frac{1}{4}(3\alpha' - 2)^2 - 1$, if α' is even, and $\frac{3}{4}(\alpha' - 1)(3\alpha' - 1) - 1$, if α' is odd; α' cannot take the value unity in this case, since $\alpha' = \frac{1}{3}(p - q + 4) = 1$ requires that $p - q = -1$, which is impossible, p always being at least as great as q in the geometry on Σ.

The general equation $\phi = 0$, for any possible values of p and q, where $p - q \equiv 2 \pmod 3$, has been found to represent a curve (p, q) of order $p + 2q$, cut from Σ by a surface S of order $\frac{1}{3}(p + 2q + 4)$, the residual intersection consisting of the cuspidal edge of Σ occurring 4 times, while $f = 2$, $R = 4$, $R_i = 4$, $R_c = 0$. If $\alpha' = 1$, the equation of the resultant surface of intersection S' must be of degree as great as unity in x, y, since the equation of the surface S was found to be of degree two in x, y, and the factor $x^{\alpha'} = x$ only has been rejected to obtain the former equation from the latter; the coefficient $a_{0,0} = 0$, in accordance with the conditions imposed here, and that is the only coefficient made to vanish in this case; some term in the variables μ, ν of the form $a_{0,\gamma}\mu^{p-q+\gamma}\nu^{q-\gamma}$, where $\gamma \geqq 1$, must appear, otherwise the equation of the curve $(p, q)_1$ would be reducible at once by λ; the term $a_{0,\gamma}\mu^{p-q+\gamma}\nu^{q-\gamma}$, under the required substitutions, takes the form

$$a_{0,\gamma}\, x^{\frac{2}{3}(p-q+\gamma)} z^{\frac{1}{3}(p-q+\gamma)} s^{q-\gamma}, \qquad \text{if } p - q + \gamma \equiv 0 \pmod 3,$$

$$a_{0,\gamma}\, x^{\frac{2}{3}(p-q+\gamma-1)} y z^{\frac{1}{3}(p-q+\gamma-1)} s^{q-\gamma}, \quad \text{if } p - q + \gamma \equiv 1 \pmod 3,$$

or

$$a_{0,\gamma}\, x^{\frac{2}{3}(p-q+\gamma-2)} y^2 z^{\frac{1}{3}(p-q+\gamma-2)} s^{q-\gamma}, \quad \text{if } p - q + \gamma \equiv 2 \pmod 3.$$

The condition that $p - q \equiv 2 \pmod 3$ demands that γ take a value at least as great as two in some term of the equation; $\gamma = 2$ makes $p - q + \gamma \equiv 1 \pmod 3$, giving a term $a_{0,2}x^{\frac{2}{3}(p-q+1)} yz^{\frac{1}{3}(p-q+1)} s^{q-2}$; the rejection of the factor $x^{\alpha+\alpha'} = x^{\frac{2}{3}(p-q+1)}$ leaves this term of degree unity in x, y and containing the variable y; hence the surface S' contains the cuspidal edge of Σ twice and only twice; thus the residual intersection of Σ and S' is made up in this case of the cuspidal edge of Σ occurring twice; consequently, $m' = \frac{1}{3}(p + 2q + 4) - 1 = \frac{1}{3}(p + 2q + 1)$ and $m = p + 2q - 1$, while $f = 1$, $R = 2$, $R_i = 2$, $R_c = 0$. The curve (p, q) was found in the cases in question to have $\pi = p - q$; that curve has now been broken up into the curve $(p, q)_1$ and the cuspidal edge occurring once, since the total order in each case is $p + 2q$; hence the curve $(p, q)_1$ has $\pi = p - q - 1$,

and, similarly, $\rho = 2q - 1$. Curves $(p, q)_1$, which are none of them curves of total intersection, when $p - q \equiv 2 \pmod 3$, will be grouped as Class IV″, and may be characterized thus:

IV″. $p - q \equiv 2 \pmod 3$, $\alpha' = 1$, $m = p + 2q - 1$, $\pi = p - q - 1$,
$\rho = 2q - 1$, $m' = \frac{1}{3}(p + 2q + 1)$, $f = 1$, $R = 2$, $R_i = 2$, $R_c = 0$,

If $\alpha' \geqq 2$, the surface S' must be of order less by α' than the order of S, i. e. $m' = \frac{1}{3}(p + 2q + 4) - \alpha'$. As in the preceding cases, the curve $(p, q)_{\alpha'}$ will be in general, a proper curve, and the entire residual intersection will be made up of the cuspidal edge of Σ occurring the necessary number of times. The coefficient $a_{\beta, \gamma}$, where $\beta = 3\alpha' - 2\gamma - 4$ and $p - q + \gamma - \beta = p - q + 3\gamma - 3\alpha' + 4 \equiv 0 \pmod 3$ does not vanish under the conditions imposed; this condition is possible of satisfaction in every case in question, since $3\alpha' - 2\gamma - 4 \geqq 0$ so long as $\alpha' \geqq 2$; hence there occurs in the equation of the curve $(p, q)_{\alpha'}$ a term or terms of the form $a_{\beta, \gamma} \lambda^{\beta} \mu^{p - q + \gamma - \beta} \nu^{q - \gamma} = a_{3\alpha' - 2\gamma - 4, \gamma} \lambda^{3\alpha' - 2\gamma - 4} \mu^{p - q + 3\gamma - 3\alpha' + 4} \nu^{q - \gamma}$; this term becomes $a_{3\alpha' - 2\gamma - 4, \gamma} x^{3\alpha' - 2\gamma - 4 + \frac{2}{3}(p - q) + 2\gamma - 2\alpha' + \frac{8}{3}} z^{\frac{1}{3}(p - q + 4) + \gamma - \alpha'} s^{q - \gamma}$, and on rejecting $x^{\alpha + \alpha'} = x^{\frac{2}{3}(p - q - 2) + \alpha'}$, this becomes $a_{3\alpha' - 2\gamma - 4, \gamma} z^{\frac{1}{3}(p - q + 4) + \gamma - \alpha'} s^{q - \gamma}$, which involves neither x nor y; hence S' does not contain the cuspidal edge of Σ, there is no residual intersection, and consequently all curves $(p, q)_{\alpha'}$ where $\alpha' \geqq 2$ and $p - q \equiv 2 \pmod 3$, are curves of total intersection of Σ and a surface S'. The curve $(p, q)_{\alpha'}$ is of order $m = 3\left[\frac{1}{3}(p + 2q + 4) - \alpha'\right] = p + 2q - 3\alpha' + 4$, which is less by $3\alpha' - 4$ than the order of the curve (p, q); this reduction in order has resulted from rejecting the cuspidal edge of Σ $3\alpha' - 4$ times; hence the number of times the curve $(p, q)_{\alpha'}$ passes through the vertex of Σ must be less than in the case of (p, q) by that same number; thus it is found here that $\pi = p - q - 3\alpha' + 4$. In like manner it is seen that $\rho = 2q - 3\alpha' + 4$. Here again $\pi \equiv 0 \pmod 3$. If both π and ρ have the value zero, then must the curve $(p, q)_{\alpha'}$ contain neither the vertex of Σ nor the cusp of the infinite cubic, and hence is a more general curve than a (p, q) of the same order, which must always contain one or both of the two points in question. Such a curve $(p, q)_{\alpha'}$ is given when $\pi = p - q - 3\alpha' + 4 = \rho = 2q - 3\alpha' + 4 = 0$, which requires that $p = 3q$ and $\alpha' = \frac{2}{3}q + \frac{4}{3} = \frac{2}{9}p + \frac{4}{3}$, while $m = p$, $m' = \frac{1}{3}p$, and $2q = p - q \equiv 2 \pmod 3$, making $q \equiv 1 \pmod 3$; the $(3, 1)_1$ answering these conditions,

—that $\alpha' = 2$, etc.,—is cut from Σ by the surface S' whose equation is found as follows: The most general equation of a curve (3, 1) is of the form

$$a_{3,1}\lambda^3 + a_{2,1}\lambda^2\mu + a_{1,1}\lambda\mu^2 + a_{0,1}\mu^3 + (a_{2,0}\lambda^2 + a_{1,0}\lambda\mu + a_{0,0}\mu^2)\nu = 0\,.$$

The conditions to be imposed on the coefficients of this equation, in order that α' may take its greatest value two, are found by the method used above to be in number $\frac{1}{2}(3\alpha' - 2)^2 - 1 = \frac{1}{4}(6-2)^2 - 1 = 3$; these conditions are satisfied by making $a_{1,1} = a_{1,0} = a_{0,0} = 0$, the other coefficients remaining; this leaves the equation of the curve $(3, 1)_1$, which is then of the form

$$a_{3,1}\lambda^3 + a_{2,1}\lambda^2\mu + a_{0,1}\mu^3 + a_{2,0}\lambda^2\nu = 0\,.$$

Performing the required substitution, this becomes

$$a_{3,1}x^3 + a_{2,1}x^2y + a_{0,1}y^3 + a_{2,0}x^2s = 0\,,$$

and the change of y^3 to x^2z allows the rejection of the factor x^2, so that the equation of the surface S' is found to be

$$a_{3,1}x + a_{2,1}y + a_{0,1}z + a_{2,0}s = 0,$$

the most general equation of the plane; hence the curve $(3,1)_1$ is the most general curve which can be cut from Σ by a plane.

Similarly, the $(12,4)_4$ of order 12 is the total intersection of Σ with the general quartic surface; the $(21,7)_6$, a curve of order 21, is the total intersection of Σ with the general surface of order 7; the $(30,10)_8$, a curve of order 30, is the total intersection of Σ with the general surface of order 10; etc., etc.

These special curves $(p.\,q)_{\alpha'}$ of total intersection will form Class IV$'$ and may be characterized thus:

IV$'$. $p - q \equiv 2 \pmod 3$, $\alpha' \geqq 2$, $m = p + 2q - 3\alpha' + 4$, $\pi = p - q - 3\alpha' + 4$,
$\rho = 2q - 3\alpha' + 4$, $m' = \frac{1}{3}(p + 2q + 4) - \alpha'$, $f = R = R_i = R_c = 0$.

Letting C denote the number of conditions imposed on the equation of general form to obtain the restricted equation $\phi = 0$, the results derived in the preceding pages may be collected and arranged thus:

Curves (p, q) Given by Equations of General Form.

1. Curves of Total Intersection.

I. $p = q$, $m = 3p$, $\pi = 0$, $\rho = 2q = 2p$, $m' = p$, $f = R = R_i = R_c = 0$.

2. Curves of Partial Intersection.

II. $p > q$, $p - q \equiv 0 \pmod 3$, $m = p + 2q$, $\pi = p - q$, $\rho = 2q$,
$m' = \frac{1}{3}(p + 2q) + 1$, $f = 1$, $R = 3$, $R_i = 2$, $R_c = 1$.

III. $p - q \equiv 1 \pmod 3$, $m = p + 2q$, $\pi = p - q$, $\rho = 2q$,
$m' = \frac{1}{3}(p + 2q + 2)$, $f = 1$, $R = 2$, $R_i = 2$, $R_c = 0$.

IV. $p - q \equiv 2 \pmod 3$, $m = p + 2q$, $\pi = p - q$, $\rho = 2q$,
$m' = \frac{1}{3}(p + 2q + 4)$, $f = 2$, $R = 4$, $R_i = 4$, $R_c = 0$.

Curves (p, q) Given by Equations of Special Form.

1. Curves of Total Intersection.

II′. $p > q$, $p - q \equiv 0 \pmod 3$, $\alpha' \geqq 1$, $m = p + 2q - 3(\alpha' - 1)$,
$\pi = p - q - 3(\alpha' - 1)$, $\rho = 2q - 3(\alpha' - 1)$, $m' = \frac{1}{3}(p + 2q) - \alpha' + 1$,
$f = R = R_i = R_c = 0$; here $\alpha' \leqq \frac{1}{3}(p - q) + 1$,

and also
$$\alpha' \leqq \begin{cases} \frac{2}{3}q + 1, & \text{if } p \equiv 0 \pmod 3, \\ \frac{2}{3}q + \frac{1}{3}, & \text{if } p \equiv 1 \pmod 3, \\ \frac{2}{3}q - \frac{1}{3}, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

and
$$C = \begin{cases} \frac{3}{4}\alpha'(3\alpha' - 2) + 1, & \text{if } \alpha' < \frac{1}{3}(p - q) + 1 \text{ and even}, \\ \frac{1}{4}(3\alpha' - 1)^2, & \text{if } \alpha' < \frac{1}{3}(\nu - q) + 1 \text{ and odd}, \\ \text{or} & \text{if } \alpha' = \frac{1}{3}(p - q) + 1 = 1, \\ \frac{3}{4}\alpha'(3\alpha' - 2), & \text{if } \alpha' = \frac{1}{3}(p - q) + 1 \text{ and even}, \\ \frac{1}{4}(3\alpha' - 1)^2 - 1, & \text{if } \alpha' = \frac{1}{3}(p - q) + 1 > 1 \text{ and odd}. \end{cases}$$

III′. $p - q \equiv 1 \pmod 3$, $\alpha' \geqq 1$, $m = p + 2q - 3\alpha' + 2$, $\pi = p - q - 3\alpha' + 2$,
$\rho = 2q - 3\alpha' + 2$, $m' = \frac{1}{3}(p + 2q + 2) - \alpha'$, $f = R = R_i = R_c = 0$; here
$\alpha' \leqq \frac{1}{3}(p - q + 2)$,

and also
$$\alpha' \leqq \begin{cases} \frac{2}{3}q + \frac{2}{3}, & \text{if } p \equiv 0 \pmod 3, \\ \frac{2}{3}q \;\;, & \text{if } p \equiv 1 \pmod 3, \\ \frac{2}{3}q - \frac{2}{3}, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

and

$$C = \begin{cases} (\tfrac{3}{2}\alpha')^2, & \text{if } \alpha' < \tfrac{1}{3}(p-q+2) \text{ and even,} \\ \tfrac{1}{4}(9\alpha'^2 - 1), & \text{if } \alpha' < \tfrac{1}{3}(p-q+2) \text{ and odd,} \\ \text{or} & \text{if } \alpha' = \tfrac{1}{3}(p-q+2) = 1, \\ (\tfrac{3}{2}\alpha')^2 - 1, & \text{if } \alpha' = \tfrac{1}{3}(p-q+2) \text{ and even,} \\ \tfrac{1}{4}(9\alpha'^2 - 1) - 1, & \text{if } \alpha' = \tfrac{1}{3}(p-q+2) \text{ and odd.} \end{cases}$$

IV′. $p - q \equiv 2 \pmod 3$, $\alpha' \geq 2$, $m = p + 2q - 3\alpha' + 4$, $\pi = p - q - 3\alpha' + 4$ $\rho = 2q - 3\alpha' + 4$, $m' = \frac{1}{3}(p + 2q + 4) - \alpha'$, $f = R = R_i = R_c = 0$; here $\alpha' \leqq \frac{1}{3}(p - q + 4)$,

and also

$$\alpha' \leqq \begin{cases} \tfrac{2}{3}q + \tfrac{4}{3}, & \text{if } p \equiv 0 \pmod 3, \\ \tfrac{2}{3}q + \tfrac{2}{3}, & \text{if } p \equiv 1 \pmod 3, \\ \tfrac{2}{3}q \quad, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

and

$$C = \begin{cases} \tfrac{1}{4}(3\alpha' - 2)^2, & \text{if } \alpha' < \tfrac{1}{3}(p-q+4) \text{ and even,} \\ \tfrac{3}{4}(\alpha' - 1)(3\alpha' - 1) & \text{if } \alpha' < \tfrac{1}{3}(p-q+4) \text{ and odd,} \\ \tfrac{1}{4}(3\alpha' - 2)^2 - 1, & \text{if } \alpha' = \tfrac{1}{3}(p-q+4) \text{ and even,} \\ \tfrac{3}{4}(\alpha' - 1)(3\alpha' - 1), & \text{if } \alpha' = \tfrac{1}{3}(p-q+4) \text{ and odd} \\ \quad \text{(the case where } \alpha' = \tfrac{1}{3}(p-q+4) = 1 \text{ coming under IV}''\text{).} \end{cases}$$

2. Curves of Partial Intersection.

IV″. $p - q \equiv 2 \pmod 3$, $\alpha' = 1$, $m = p + 2q - 1$, $\pi = p - q - 1$, $\rho = 2q - 1$, $m' = \frac{1}{3}(p + 2q + 1)$, $f = 1$, $R = 2$, $R_i = 2$, $R_c = 0$

and $C = 1$.

The following table gives the results found by the formulæ given above for the cases of all proper curves (p, q) for values of p and q as far as the curve $(8, 6)$.

TABLE.

p	q	$p-q$	a'	C	m	π	ρ	Class	m'	f	R	R_i	R_c
1	0	1	0	0	1	1	0	III.	1	1	2	2	0
1	1	0	0	0	3	0	2	I.	1	0	0	0	0
2	1	1	0	0	4	1	2	III.	2	1	2	2	0
2	2	0	0	0	6	0	4	I.	2	0	0	0	0
3	1	2	0	0	5	2	2	IV.	3	2	4	4	0
			1	1	4	1	1	IV.″	2	1	2	2	0
			2	3	3	0	0	IV.′	1	0	0	0	0
3	2	1	0	0	7	1	4	III.	3	1	2	2	0
			1	2	6	0	3	III.′	2	0	0	0	0
3	3	0	0	0	9	0	6	I.	3	0	0	0	0
4	1	3	0	0	6	3	2	II.	3	1	3	2	1
			1	1	6	3	2	II.′	2	0	0	0	0
4	2	2	0	0	8	2	4	IV.	4	2	4	4	0
			1	1	7	1	3	IV.″	3	1	2	2	0
			2	3	6	0	2	IV.′	2	0	0	0	0
4	3	1	0	0	10	1	6	III.	4	1	2	2	0
			1	2	9	0	5	III.′	3	0	0	0	0
4	4	0	0	0	12	0	6	I.	4	0	0	0	0
5	1	4	0	0	7	4	2	III.	3	1	2	2	0
5	2	3	0	0	9	3	4	II.	4	1	3	2	1
			1	1	9	3	4	II.′	3	0	0	0	0
5	3	2	0	0	11	2	6	IV.	5	2	4	4	0
			1	1	10	1	5	IV.″	4	1	2	2	0
			2	3	9	0	4	IV.′	3	0	0	0	0
5	4	1	0	0	13	1	8	III.	5	1	2	2	0
			1	2	12	0	7	III.′	4	0	0	0	0
5	5	0	0	0	15	0	10	I.	5	0	0	0	0
6	1	5	0	0	8	5	2	IV.	4	2	4	4	0
			1	1	7	4	1	IV.″	3	1	2	2	0
			2	4	6	3	0	IV.′	2	0	0	0	0
6	2	4	0	0	10	4	4	III.	4	1	2	2	0
			1	2	9	3	3	III.′	3	0	0	0	0
			2	8	6	0	0	III.′	2	0	0	0	0
6	3	3	0	0	12	3	6	II.	5	1	3	2	1
			1	1	12	3	6	II.′	4	0	0	0	0
			2	6	9	0	3	II.′	4	0	0	0	0
6	4	2	0	0	14	2	8	IV.	6	2	4	4	0
			1	1	13	1	7	IV.″	5	1	2	2	0
			2	3	12	0	6	IV.′	4	0	0	0	0
6	5	1	0	0	16	1	10	III.	6	1	2	2	0
			1	2	15	0	9	III.′	5	0	0	0	0
6	6	0	0	0	18	0	12	I.	6	0	0	0	0
7	1	6	0	0	9	6	2	II.	4	1	3	2	1
			1	1	9	6	2	II.′	3	0	0	0	0
7	2	5	0	0	11	5	4	IV.	5	2	4	4	0
			1	1	10	4	3	IV.″	4	1	2	2	0
			2	4	9	3	2	IV.′	3	0	0	0	0
7	3	4	0	0	13	4	6	III.	5	1	2	2	0
			1	2	12	3	5	III.′	4	0	0	0	0
			2	8	9	0	2	III.′	3	0	0	0	0
7	4	3	0	0	15	3	8	II.	6	1	3	2	1
			1	1	15	3	8	II.′	5	0	0	0	0
			2	6	12	0	5	II.′	4	0	0	0	0
7	5	2	0	0	17	2	10	IV.	7	2	4	4	0
			1	1	16	1	9	IV.″	6	1	2	2	0
			2	3	15	0	8	IV.′	5	0	0	0	0
7	6	1	0	0	19	1	12	III.	7	1	2	2	0
			1	2	18	0	11	III.′	6	0	0	0	0
7	7	0	0	0	21	0	14	I.	7	0	0	0	0
8	1	7	0	0	10	7	2	III.	4	1	2	2	0
8	2	6	0	0	12	6	4	II.′	4	0	0	0	0
			1	1	12	6	4	II.′	4	0	0	0	0
8	3	5	0	0	14	5	6	IV.	6	2	4	4	0
			1	1	13	4	5	IV.″	5	1	2	2	0
			2	4	12	3	4	IV.′	4	0	0	0	0
8	4	4	0	0	16	4	8	III.	6	1	2	2	0
			1	2	15	3	7	III.′	5	0	0	0	0
			2	8	12	0	4	III.′	4	0	0	0	0
8	5	3	0	0	18	3	10	II.	7	1	3	2	1
			1	1	18	3	10	II.′	6	0	0	0	0
			2	6	15	0	7	II.′	5	0	0	0	0
8	6	2	0	0	20	2	12	IV.	8	2	4	4	0

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The Plane Geometry of the Point in Point-Space of Four Dimensions.

By C. J. Keyser.

I.—*Introductory Considerations.*

1. As is well known, the dimensionality (in Riemann's sense) of any given space depends upon the element chosen for its construction; and in accordance with the Plücker principle of counting constants, any given space may be made to assume any prescribed dimensionality k by merely taking for element a configuration for whose determination within that space k independent data are necessary and sufficient—a configuration, in other words, whose general analytical representation in the given space involves exactly k parameters. A space being assumed, there are, in general, infinitely many possible choices of element for which the space will have a previously assigned dimensionality. Of such possible choices the great majority would be inexpedient as not leading to interesting results. Of all elements, in case of any given space, those are, in general, most practicable which present themselves in *pairs of reciprocals*, as in the familiar examples of the point and line in the plane, the line and plane in the sheaf, and the point and plane in ordinary space.

A space that is n-dimensional in points is also n-dimensional in point-spaces of $n - 1$ dimensions. It has $2(n - 1)$ dimensions both in lines and in point-spaces of $n - 2$ dimensions; and, in general, its dimensionality is $p(n - p + 1)$ if the point-space either of $p - 1$ or of $n - p$ dimensions be taken as element. Not only, however, do the two last mentioned elements furnish the same dimensionality, which is a necessary though not a sufficient condition for reciprocity, but they are indeed reciprocal elements in n-fold point-space; for the same system of equations, which on proper interpretation defines one of the elements, admits a second (dual) interpretation defining the other. It thus appears that by taking as elements the various point-spaces of less than n dimensions for the construc-

tion of n-fold point-space, there arise n geometries of this space; or, if we regard two reciprocal theories as but two phases of one geometry, the elements in question yield $\frac{n}{2}$ or $\frac{n-1}{2}+1$ geometries according as n is even or odd, the element having $\frac{n-1}{2}$ dimensions being, in case of n odd, its own reciprocal, or *self-reciprocal.*

Like considerations hold for spaces of n dimensions in other elements than points. It will be convenient, however, and sufficient to conduct this discussion for space supposed n-fold in points.

2. Of such geometries the self-reciprocal, or those arising from the use of self-reciprocal elements, are of special interest as well from the artistic as from the scientific point of view. The precise nature of the distinction in question may be made sufficiently clear by the following considerations. In n-fold space a definite configuration C, including this space itself as a special case, may, in general, be regarded at will as an assemblage of points or of lines or of planes and so on up to $\overline{n-1}$-fold point-spaces. These n assemblages, which may be denoted respectively by $E_0, E_1, \ldots, E_{n-1}$, the subscripts indicating the point dimensionality of the elements of the corresponding assemblages, are equivalent not only in the assemblage theory sense of the term but also in the logical sense that they serve as so many distinct definitions or conceptions of one and the same configuration. While distinct, they are of course not independent. If, for example, C be supposed to represent a curve of $\overline{n-1}$-ple curvature, E_0 will naturally be the assemblage of its points, E_1 the assemblage of its tangent lines, E_2 that of its osculating planes, ..., and the E's are accordingly to be thought as having a one-parameter dependence, by virtue of which to each element of C belonging to one E, there corresponds in general one and but one element of C belonging to each other E. Now, under a homographic transformation, the n E's are converted into n other assemblages $E'_0, E'_1, \ldots, E'_{n-1}$ in such a way that any E and the corresponding E' are of the same kind, have, i. e., the same subscript. The E''s are connected like the E's and in their turn serve as n distinct definitions of one and the same configuration C', the transformed of C. We may say, then, that each of the indicated definitions or conceptions of a given configuration is preserved in kind under a homographic transformation. Such is,

however, in general, not the case under a dualistic transformation; for, while the latter converts the n E's into n E''s, of which each serves as a definition of the transformed configuration C' of C, any E and the corresponding E' are, in general, not of a kind; if the subscript of the former be k, that of the latter will be $n-k-1$, and these cannot be equal unless n be odd, and in this case only for a single value of k, namely, $k=\overline{n-1}:2$, n being given. This case excepted, no definition of C is preserved under dualistic transformation; the point, line, ..., conceptions of C pass over respectively into the $\overline{n-1}$-space, $\overline{n-2}$-space, ..., conceptions of C'; but in the case where n is odd and equal (say) to $2m+1$, the assemblage E_m defining C is converted into an assemblage E'_m defining C'; the conception is preserved in kind. Now when $n=2m+1$, the elements of E_m are self-reciprocal elements of n-fold space, and under no other circumstances are the elements of any E self-reciprocal. We arrive accordingly at this conclusion: *The distinction of the self-reciprocal geometries among other geometries is the definitional or conceptual invariance,* in case of the former, of all configurations, under both the homographic and the dualistic transformations.* Because of this property of invariance, one may say that the m-space conception of configurations in space of $2m+1$ dimensions is of higher scientific value, as being more central and penetrating, in more perfect accord with the intimate nature of space itself, than are such conceptions as lose their identity under one or the other of the mentioned modes of transformations—an estimate, moreover, that seems to be justified by the highly artistic analytical form which self-reciprocal theories are, it is well known, capable of assuming.

3. Point-space of 4 dimensions is also 4-dimensional in ordinary 3-dimensional spaces, or *lineoids*,† the point and the lineoid being reciprocal elements. It is 6-dimensional in lines and in planes, which are also reciprocal elements. This space contains no linear self-reciprocal element and admits of no self-reciprocal construction. Nevertheless there are two self-reciprocal theories of spaces (the point and the lineoid) *within* 4-fold point space, which, besides their own intrinsic

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* It is interesting that the significance of this property, which was pointed out and made a principle of procedure in the line geometry of ordinary space by Klein, Koenigs and others, seems not to have been fully appreciated by Plücker, whose method even in line theory never became quite free from the relatively cumbrous point-plane conception of space.

† Cf. Cole: "On Rotations in Space of Four Dimensions." Amer. Math. Journ., Vol. 12.

interest, are of the greatest importance in building up as well the point-lineoid as the line-plane geometry of 4-fold space itself. Just as any *lineoid* of this space is 3-dimensional in points and in planes and 4-dimensional in *lines*, so any *point* of the same space is 3-dimensional in lineoids and in lines and 4-dimensional in *planes;* and just as the line geometry (the Plücker theory) of a lineoid, regarded as a space of lines, is a self-reciprocal geometry, so the plane geometry of a point, regarded as a space or plenum of planes containing it, is a self-reciprocal theory. With the evidently possible parallelization of these coordinate self-reciprocal theories with the point-lineoid geometry of 4-space, we are not here concerned. Our interest lies in the theories as such, in their relations with one another and in the completely correlative rôles they play particularly in the development of the *line-plane* geometry of 4-space. The line geometry of the lineoid has been often treated and is familiar enough, at least in its elements. On the other hand, the plane geometry of the point (in 4-space) has not, so far as we are aware, been systematically developed.* This paper undertakes to construct so much of this theory as in connection with the other will be of immediate service in investigating the line-plane geometry of 4-space, to which subject this article is intended as a preliminary contribution.

II.—*Homogeneous Coordinates of the Plane.*

4. We enter here directly upon the subject proper of this paper: the plane theory of the point in 4-space. The space with which we have to deal is the point regarded as the assemblage of all the lineoids, planes and lines of 4-space that pass through (or contain) it. As the plane is to be taken as element, the point will be for us primarily a space of planes. This hypersheaf will be supposed given once for all, and, except where the contrary is indicated, all lines, planes and lineoids considered will be supposed to belong to it.

Let the assumed point be given by the four lineoids

$$A_1 = 0, \quad A_2 = 0, \quad A_3 = 0, \quad A_4 = 0,$$

where

$$A_k \equiv a_5^{(k)} + \Sigma a_i^{(k)} X_i \qquad (i = 1, 2, 3, 4).$$

The assemblage of generating lineoids may be represented by the equation

$$x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 = 0,$$

* In the cited article by Cole several theorems of the present paper are established.

where the x_i are parameters. Each system of values of x_i defines a lineoid and to each lineoid corresponds a unique system of values of the ratios $x_1 : x_2 : x_3 : x_4$. A linear equation

$$\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 = 0 \equiv \Sigma \xi_i x_i \tag{1}$$

where the ξ_i are supposed given, will define a *line* ξ_i as an *envelope* of *lineoids*. On the other hand if the ξ_i be regarded as variable and the x_i as given, the same equation will define a *lineoid* x_i as *locus* of *lines*, a bundle. Accordingly a pair of equations

$$\Sigma \xi_i x_i = 0, \quad \Sigma \xi_i' x_i = 0$$

will represent a *plane* (ξ_i, ξ_i') as an *envelope* of *lineoids*, while a pair

$$\Sigma x_i \xi_i = 0, \quad \Sigma x_i' \xi_i = 0$$

will represent a *plane* (x_i, x_i') as a *locus* of *lines*, a *flat pencil*.

We will employ x_i and ξ_i respectively as associated homogeneous lineoid and line coordinates.* The configuration of common reference will be that composed of the four lineoids A_1, A_2, A_3, A_4, the six planes A_1A_2, A_1A_3, A_1A_4, A_2A_3, A_2A_4, A_3A_4, and the four lines $A_1A_2A_3$, $A_1A_2A_4$, $A_1A_3A_4$, $A_2A_3A_4$. The coordinate lineoids will be represented in line coordinates by $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, $\xi_4 = 0$, and in lineoid coordinates by $x_2 = x_3 = x_4 = 0$, $x_1 = x_3 = x_4 = 0$, $x_1 = x_2 = x_4 = 0$, $x_1 = x_2 = x_3 = 0$; the fundamental lines will be represented by $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, or by $\xi_2 = \xi_3 = \xi_4 = 0$, $\xi_1 = \xi_3 = \xi_4 = 0$, $\xi_1 = \xi_2 = \xi_4 = 0$, $\xi_1 = \xi_2 = \xi_3 = 0$; while the planes of reference will be given by $x_1 = x_2 = 0$, $x_1 = x_3 = 0$, $x_1 = x_4 = 0$, $x_2 = x_3 = 0$, $x_2 = x_4 = 0$, $x_3 = x_4 = 0$, or by $\xi_3 = \xi_4 = 0$, $\xi_2 = \xi_4 = 0$, $\xi_2 = \xi_3 = 0$, $\xi_1 = \xi_4 = 0$, $\xi_1 = \xi_3 = 0$, $\xi_1 = \xi_2 = 0$. The equation (1) also signifies that the line ξ_i and the lineoid x_i are *united in position*, *i. e.*, that the *line lies in the lineoid* and the *lineoid contains the line*.

5. As already indicated, any plane whatever may appear in either of two aspects: as a *locus* of its *lines*, *i. e.*, as a *flat pencil*, or as an *envelope* of its generating *lineoids*, *i. e.*, the lineoids containing the plane. These dual conceptions of the plane correspond precisely, in the order named, to the two Plücker

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***Such systems might legitimately have been assumed immediately by virt. e of the projective correspondence, already noticed, between the elements of the hypersheaf under investigation and the elements of the lineoid, according to which the lines, planes and lineoids of the one assemblage correspond respectively in a one-to-one way to the planes, lines and points of the other.**

conceptions of the line (in ordinary space), namely, axis and ray (*Axe, Strahl*). The plane, being geometrically determined as a flat pencil by any two of its lines, is determined analytically by two sets of line coordinates; while, being geometrically determined as an envelope by any two of its (generating) lineoids, it is determined analytically by two systems of lineoid coordinates. As explained below, the first two sets when combined will furnish one system of homogeneous plane coordinates and the second two sets similarly combined will yield a second system.

Consider any two lines ξ_i and η_i $(i = 1, 2, 3, 4)$. These determine a plane π, which is equally determined by any two lines of the pencil

$$\Sigma(\lambda\xi_i x_i + \mu\eta_i x_i) = 0, \qquad (i = 1, 2, 3, 4), \tag{2}$$

and in particular by any two of the four special lines obtained by equating successively to zero the coefficients of x_1, x_2, x_3, x_4 in (2),

$$\left.\begin{aligned} (\xi_1\eta_2 - \xi_2\eta_1)x_2 + (\xi_1\eta_3 - \xi_3\eta_1)x_3 + (\xi_1\eta_4 - \xi_4\eta_1)x_4 = 0,\\ -(\xi_1\eta_2 - \xi_2\eta_1)x_1 + (\xi_2\eta_3 - \xi_3\eta_2)x_3 + (\xi_2\eta_4 - \xi_4\eta_2)x_4 = 0,\\ -(\xi_1\eta_3 - \xi_3\eta_1)x_1 - (\xi_2\eta_3 - \xi_3\eta_2)x_2 + (\xi_3\eta_4 - \xi_4\eta_3)x_4 = 0,\\ -(\xi_1\eta_4 - \xi_4\eta_1)x_1 - (\xi_2\eta_4 - \xi_4\eta_2)x_2 - (\xi_3\eta_4 - \xi_3\eta_4)x_3 = 0. \end{aligned}\right\} \tag{3}$$

These are the four lines of the pencil (2) *that lie one in each of the fundamental lineoids.* The ratios of the coefficients of any two of the lines furnish the four constants upon which the position of π depends The choice of two of the lines would, however, be arbitrary, and we may avoid such choice, while at the same time securing symmetry, by retaining for coordinates of π the entire six coefficients of equations (3). These, again, may be replaced by an arbitrary multiple of them, since only their ratios are material. We will accordingly have for *coordinates of the plane regarded as a flat pencil* the six quantities

$$\rho p_{ik} = \xi_i\eta_k - \xi_k\eta_i. \tag{4}$$

On expanding the identically vanishing determinant

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{vmatrix} = \Delta,$$

in terms of quadratic minors, we find that the six coordinates are connected

by the identity

$$\tfrac{1}{2}\omega(p) \equiv 0 = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23}, \tag{5}$$

showing, as ought to be the case, that the 5 ratios of the 6 p's are equivalent to but 4 independents. It can be readily shown that any six quantities p_{ik} satisfying the identity (5) and being such that $p_{ik} = -p_{ki}$, serve to determine a plane (or pencil), and that if ξ_i' and η_i' be any two lines of the pencil, $p_{ik} : p_{ik}' = k$, a constant.

5. To find a corresponding *system of coordinates for the plane conceived as an envelope of lineoids,* suppose it given by two lineoids x_i and y_i. Of the lineoids of the pencil

$$\Sigma(\lambda x_i \xi_i + \mu y_i \xi_i) = 0, \qquad (i = 1, 2, 3, 4) \tag{6}$$

of generators of π, the following

$$\begin{cases} q_{12}\xi_2 + q_{13}\xi_3 + q_{14}\xi_4 = 0, \\ q_{21}\xi_1 + q_{23}\xi_3 + q_{24}\xi_4 = 0, \\ q_{31}\xi_1 + q_{32}\xi_2 + q_{34}\xi_4 = 0, \\ q_{41}\xi_1 + q_{42}\xi_2 + q_{43}\xi_3 = 0, \end{cases}$$

where

$$\sigma q_{ik} = x_i y_k - x_k y_i \tag{8}$$

are the four generators of which each contains one and but one of the fundamental lines.

The six coefficients q_{ik} are connected by the quadratic identity

$$\tfrac{1}{2}\omega(q) \equiv 0 = q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23}, \tag{9}$$

and for reasons precisely analogous to those given for the p's, any six quantities q_{ik} satisfying the identity (9) and being such that $q_{ik} = -q_{ki}$, suffice to determine a plane uniquely, and may be taken for homogeneous coordinates of the same regarded as enveloped by lineoids.

7. Inasmuch as the configurations with which we shall be concerned are, most of them, to be conceived as assemblages of *planes*, and since the latter are self-reciprocal elements (cf. §I), there is, in general, no advantage, but often rather a disadvantage, in observing the distinction between the two aspects of the plane, as flat pencil of lines and as envelope of lineoids, in which alone the

difference between the two systems* p_{ik} and q_{ik} originates. And in fact it is easy to show that these systems, while they differ in the sense indicated, are *as* coordinates, as data fixing the position of a plane, *identical* function for function, a proportionality factor being of course excepted. To effect this identification it is sufficient to find the condition that p_{ik} and q_{ik} shall determine one and the same plane. Suppose the plane determined by p_{ik} to be that represented by equations (3). If q_{ik} give the same plane, then the latter must lie in each of the lineoids x_i, y_i, which requires

$$\left.\begin{aligned} p_{12}x_2 + p_{13}x_3 + p_{14}x_4 = 0, \\ p_{12}y_2 + p_{13}y_3 + p_{14}y_4 = 0, \end{aligned}\right\} \qquad (10)$$

$$\left.\begin{aligned} p_{21}x_1 + p_{23}x_3 + p_{24}x_4 = 0, \\ p_{21}y_1 + p_{23}y_3 + p_{24}y_4 = 0, \end{aligned}\right\} \qquad (11)$$

$$\left.\begin{aligned} p_{31}x_1 + p_{32}x_2 + p_{34}x_4 = 0, \\ p_{31}y_1 + p_{32}y_2 + p_{34}y_4 = 0, \end{aligned}\right\} \qquad (12)$$

and a third pair, not needed.

From (10) we derive

$$p_{12} : q_{34} = p_{13} : q_{42} = p_{14} : q_{23},$$

from (11)

$$p_{12} : q_{34} = p_{23} : q_{14} = p_{42} : q_{13},$$

and from (12)

$$p_{13} : q_{42} = p_{23} : q_{14} = p_{34} : q_{12}.$$

On combining we have

$$p_{12} : q_{34} = p_{13} : q_{42} = p_{14} : q_{23} = p_{34} : q_{12} = p_{42} : q_{13} = p_{23} : q_{14}, \qquad (13)\dagger$$

which shows that if the quantities p_{ik}, taken in any order, as 12, 13, 14, 34, 42, 23, determine a plane as a flat pencil, the same quantities, taken in the equally general corresponding order 34, 42, 23, 12, 13, 14, determine the same plane as an envelope of lineoids.

8. In general, two planes have no intersection, *i. e.*, no common line. The condition that they shall have a line in common, or, what is tantamount, shall

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* There are, of course, many equivalent systems of coordinates for the plane in the hypersheaf as there are for the line in ordinary space. Indeed, in the appendix to his very first paper on the latter subject Plücker presents no less than eight distinct systems. Cf. Plücker, "On a New Geometry of Space." Phil. Trans. of the R. Soc. of London, Vol. 155. Also Wiss. Abh.

† Cf. Plücker: "Neue Geometrie des Raumes," p. 4.

lie in a same lineoid, will, like the corresponding condition* for the intersection of two lines in the Plücker line theory, assume the following four forms, if the distinction of locus and envelope be observed. According as the two planes π and π' are regarded (*a*) both as flat pencils, (*b*) π as a pencil and π' as an envelope, (*c*) π as an envelope and π' as a pencil, (*d*) both as envelopes, the forms in question will be

$$p_{12}p'_{34} + p_{13}p'_{42} + p_{14}p'_{23} + p_{34}p'_{12} + p_{42}p'_{13} + p_{23}p'_{14} = 0, \qquad (a)$$

$$\Sigma p_{ik}\cdot q'_{ik} = 0, \quad \Sigma p'_{ik}\cdot q_{ik} = 0, \qquad (b), (c)$$

$$q_{12}q'_{34} + q_{13}q'_{42} + q_{14}\cdot q'_{23} + q_{34}\cdot q'_{12} + q_{42}\cdot q'_{13} + q_{23}\cdot q'_{14} = 0. \qquad (d).$$

By virtue, however, of (13) we may write for coordinates of the plane simply six quantities r_{ik}, such that $r_{ik} = -r_{ki}$ and that $\omega(r) = 0$, and then disregard the distinction of locus and envelope. The condition that two planes

$$\begin{cases} r_{12}v_2 + r_{13}v_3 + r_{14}v_4 = 0, \\ -r_{12}v_1 + r_{23}v_3 + r_{24}v_4 = 0, \end{cases}$$

$$\begin{cases} r'_{12}v_2 + r'_{13}v_3 + r'_{14}v_4 = 0, \\ -r'_{12}v_1 + r'_{23}v_3 + r'_{24}v_4 = 0, \end{cases}$$

shall lie in a same lineoid, or intersect in a line, is, then,

$$\begin{vmatrix} 0 & r_{12} & r_{13} & r_{14} \\ r_{12} & 0 & r_{23} & r_{24} \\ 0 & r'_{12} & r'_{13} & r'_{14} \\ r'_{12} & 0 & r'_{23} & r'_{24} \end{vmatrix} = 0,$$

from which by help of the conditions, $r_{ik} = -r_{ki}$ and $\omega(r) = 0$, we readily find

$$r_{12}r'_{34} + r_{13}r'_{42} + r_{14}r'_{23} + r_{34}r'_{12} + r_{42}r'_{13} + r_{23}r'_{14} = 0.$$

Writing the left member of this polar form

$$\omega(r, r')\dagger \equiv \tfrac{1}{2}\Sigma \frac{\partial\omega(r)}{\partial r_{ik}} r'_{ik} = \tfrac{1}{2}\Sigma \frac{\partial\omega(r')}{\partial r'_{ik}} r_{ik},$$

we have the fundamental proposition: *The necessary and sufficient condition that*

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* Cf. Cayley: "On the Six Coordinates of a Line," Collected Papers, Vol. VII. Klein: "Einleitung in die höhere Geometrie," Vol. 1, p. 168.

† Cf. Pasch: "Zur Theorie der linearen Complexe," Crelle, Vol. 75, p. 11. Also, Koenigs: "La geometrie reglé," Annales de La Faculte des Sciences de Toulouse, Vol. III, p. 9.

two planes r_{ik} and r'_{ik} shall have a line in common, or lie in a same lineoid, is that the polar form $\omega(r, r')$ with respect to these planes, shall vanish.

9. The coordinates r_{ik} admit of generalization. We know from the theory* of forms that the new variables v_i in the transformation

$$\rho r_{ik} = C_{ik,1} v_1 + C_{ik,2} v_2 + \dots + C_{ik,6} v_6,$$

where the modulus is not zero, are connected by a homogeneous quadratic identity $\Omega(v) = 0$, where $\Omega(v)$ is the transformed of $\omega(r)$. Moreover, the polar form $\omega(r, r')$ of $\omega(r)$ is converted by the same transformation into the polar form $\Omega(v, v')$ of $\Omega(v)$. It is well known that $\omega(r)$ regarded as a quadratic form has a non-vanishing discriminant and that it is possible to find a linear transformation which will convert this form into any quadratic form $\Omega(v)$ whose determinant does not vanish. Accordingly we may employ for homogeneous plane coordinates any six variables v_i connected by the quadratic relation $\Omega(v) = 0$, where $\Omega(v)$ has a non-zero discriminant. Hence the condition that the planes v_i and v'_i shall intersect in a line, or lie in one lineoid, is that the polar form $\Omega(v, v')$ shall vanish.

10. It is now perfectly clear, it was indeed *a priori* evident, that the theory here in process of construction and the line theory of ordinary space, while they are geometrically distinct, disparate in fact, may be made to assume one and the same analytical aspect. Accordingly three courses lie open. The theories being coordinate in rank and being correlative auxiliary instruments for the construction of the line-plane geometry of 4-space, the ideal would seem to be to develop them as such, side by side. On the other hand, as the line theory already exists in a score of presentations, one might be content to derive the plane theory from it by translation, by merely replacing the old system of ideas by the new. Again, as neither doctrine can claim logical priority as against the other, it appears to be desirable to present the new doctrine *once* on its own account, the old having been often so presented, and not as a secondary discipline derived from another. The first course is rejected as being too long; the second is scarcely shorter and offers, besides, a false perspective. The third recommends itself as a compromise, and accordingly we shall continue, as we have begun, to construct the theory in question, as self-justified, upon its own foundations,—a course which will allow occasional pauses to note correlative propositions in the corresponding line geometry.

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* Cf. Klein: Op. cit., pp. 190, 191.

III.—*Systems of Planes.—The Linear Complex of Planes.*

11. We pass to the study of systems of planes. Of such systems there are five sorts as follows: (a) the 4-*parameter* system, which is composed of all the planes of the point, or hypersheaf under investigation, and which may be regarded as the locus of a single plane π of the system, π being subject to no condition; (b) the 3-*parameter* system, or *complex*, which is defined by imposing *one* condition upon the 4-parameter system; (c) the 2-*parameter* system, or *congruence*, the assemblage defined by a pair of conditions upon the planes of the hypersheaf; (d) the 1-*parameter* system, or *configuration* or *plane series*, an assemblage defined by a 3-fold condition; (e) the *zero-parameter* system, always a finite assemblage, defined by a set of *four* conditions upon the parameters of system (a). A plane will be said to have 4, 3, 2, 1, or 0 degrees of freedom or indetermination according as it is regarded as belonging to a 4-, 3-, 2-, 1-, or 0-parameter system.

12. Two or more planes having a line in common may be called *collinear;* two or more planes contained in a same lineoid may be called *collineoidal.* An assemblage of planes that are all of them at once collinear and collineoidal is an ordinary *axal pencil* of planes. We will, however, call such a pencil a *flat axal pencil,* reserving the name *axal pencil* for the totality of planes containing a line. Denote by ν_i' and ν_i'' any two collinear, or collineoidal, planes and consider the expression

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i''. \qquad \text{(the } \lambda\text{'s arbitrary)}$$

We have by hypothesis

$$\begin{cases} \Omega(\nu') &= 0, \\ \Omega(\nu'') &= 0, \\ \Omega(\nu', \nu'') &= 0. \end{cases}$$

Also, by identity

$$\Omega(\nu) = \Omega(\lambda_1 \nu' + \lambda_2 \nu'') = \Omega(\nu')\lambda_1^2 + \Omega(\nu'')\lambda_2^2 + 2\Omega(\nu', \nu'')\lambda_1\lambda_2,$$

whence

$$\Omega(\nu) = 0,$$

i. e., the quantities ν_i are the coordinates of a plane for all values of λ_1 and λ_2. If π_i be any plane whatever having a lineoid in common with each of the planes ν_i' and ν_i'',

$$\Omega(\nu', \pi) = 0, \quad \Omega(\nu'', \pi) = 0,$$

and therefore

$$\Omega(\nu, \pi) \equiv \Omega(\nu', \pi)\lambda_1 + \Omega(\nu'', \pi)\lambda_2 = 0,$$

i. e., the planes ν_i are collineoidal with π_i and they consequently contain the common line of ν_i' and ν_i''. It is likewise plain that the planes ν_i are all contained in the common lineoid of ν_i' and ν_i''. The planes ν_i are therefore all found in the flat axal pencil (ν_i', ν_i''). Is the converse true? Is every plane of the pencil one of the planes ν_i? Suppose ν_i''' to be an arbitrarily chosen plane of the pencil and let π_i' be any plane collineoidal with ν_i''' but not with any other plane of the pencil. Let ν_i'''' be that one of the planes ν_i for which

$$\Omega(\nu, \pi') = \Omega(\nu', \pi')\lambda_1 + \Omega(\nu'', \pi')\lambda_2 = 0.$$

The planes ν_i''' and ν_i'''' are identical. We see, therefore, that the planes ν_i constitute the flat axal pencil (ν_i', ν_i''). Accordingly, any two *collineoidal planes* ν_i' *and* ν_i'' *determine a flat axal pencil and the coordinates of the planes of the pencil are of the form*

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i''.$$

This last is identical with the form giving in ordinary space the line coordinates of the lines of a flat pencil determined by two concurrent lines.

13. As a plane of a flat axal pencil has one degree of freedom and that of a complex three degrees, a plane that belongs to both will have zero degrees of freedom, being subject to four conditions. The number of planes of a given complex that belong to an arbitrary flat axal pencil is, therefore, finite. This number will be called the *degree* of the given complex.

The assemblage of planes having a line in common—the axal pencil proper—and the assemblage of planes contained in a lineoid—the ordinary bundle of planes—are the analogues respectively of the sheaf and the plane of lines in ordinary space. The two assemblages in question, *i. e.*, the axal pencil and the bundle, being each bi-dimensional, may with propriety receive a common name. Following a suggestion of Koenigs, we will adopt for such common designation the term *hyperpencil* of planes.

14. We may now prove that a *hyperpencil of planes is completely determined by any three planes* ν_i', ν_i'', ν_i''', *such that each is collineoidal* (*or collinear*) *with each of the other two, and that all and only the planes of the hyperpencil are given by coordinates of the form*

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i'' + \lambda_3 \nu_i'''.$$

Two cases may arise. The three given planes may determine three distinct lines and one lineoid containing them or three distinct lineoids and one line contained in them. In the former case the planes are the faces of an ordinary trieder and the hyperpencil will be a bundle. We will conduct the argument for the second case, for which the hyperpencil will be an axal pencil, the proof being identical in form for both cases. By hypothesis, we have

$$\Omega(\nu')=0,\quad \Omega(\nu'')=0,\quad \Omega(\nu''')=0,$$
$$\Omega(\nu'',\nu''')=0,\quad \Omega(\nu''',\nu')=0,\quad \Omega(\nu',\nu'')=0,$$

from which it follows that

$$\begin{aligned}\Omega(\nu)&\equiv\Omega(\lambda_1\nu'+\lambda_2\nu''+\lambda_3\nu''')\\ &\equiv\Omega(\nu')\lambda_1^2+\Omega(\nu'')\lambda_2^2+\Omega(\nu''')\lambda_3^2+2\Omega(\nu'',\nu''')\lambda_2\lambda_3\\ &\qquad+2\Omega(\nu''',\nu')\lambda_3\lambda_1+2\Omega(\nu',\nu'')\lambda_1\lambda_2=0.\end{aligned}$$

Hence for every system of values of the ratios $\lambda_1:\lambda_2:\lambda_3$, the six quantities ν_i determine a plane. Now let π_i be an arbitrary plane collineoidal (or collinear) with each of the given planes ν_i', ν_i'', ν_i'''. Then

$$\Omega(\pi)=0,\quad \Omega(\nu',\pi)=0,\quad \Omega(\nu'',\pi)=0,\quad \Omega(\nu''',\pi)=0.$$

Consequently

$$\Omega(\nu,\pi)\equiv\Omega(\lambda_1\nu'+\lambda_2\nu''+\lambda_3\nu''')\equiv\Omega(\nu',\pi)\lambda_1+\Omega(\nu'',\pi)\lambda_2+\Omega(\nu''',\pi)\lambda_3=0,$$

i. e., every ν_i is collineoidal with every π_i and hence contains the line (ν',ν'',ν'''). Conversely, every plane ν_i'''' containing this line is one of the planes ν_i. For let π_i' and π_i'' be any two planes each collineoidal with but not belonging to the axal pencil. Only one plane ν_i'''' is collineoidal with each of the planes π_i' and π_i''. We prove that one of the planes ν_i is so collineoidal, whence it follows that this ν_i is identical with ν_i''''. The proof consists in showing that

$$\Omega(\nu,\pi')=0,\quad \Omega(\nu,\pi'')=0.$$

Now

$$\Omega(\nu,\pi')=\Omega(\nu',\pi')\lambda_1+\Omega(\nu'',\pi')\lambda_2+\Omega(\nu''',\pi')\lambda_3,$$
$$\Omega(\nu,\pi'')=\Omega(\nu',\pi'')\lambda_1+\Omega(\nu'',\pi'')\lambda_2+\Omega(\nu''',\pi'')\lambda_3,$$

which may both be made to vanish by a proper choice of values of the ratios of the λ's. Hence the planes ν_i constitute the planes of the axal pencil determined by the collinear planes ν_i', ν_i'', ν_i'''. In like manner, if ν_i', ν_i'', ν_i''' be the

faces of an ordinary trieder, they determine a bundle whose planes are given by the formula

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i'' + \lambda_3 \nu_i'''.$$

15. A plane that is required to belong at once to a complex and a hyperpencil, being subject to three conditions, has one degree of freedom. The locus of such a plane is, therefore, a "configuration." We will name it a cone C_1, or a cone C_2 of the complex according as the hyperpencil is an axal pencil or a bundle. C_1 and C_2 correspond precisely and respectively to the curve and the cone of a complex of lines in ordinary space. Just as the curve has all its lines in a bi-dimensional *point* manifold, the *plane*, so C_1 has its planes in a bi-dimensional *lineoid* manifold, the *line;* and just as the (line) cone has all its lines joined by a point while their points require for their representation a 3-fold manifold of points, *ordinary space* (a *lineoid*), so C_2 has all its planes in a lineoid while their (generating) lineoids require for their construction a 3-fold manifold of lineoids, a *point;* and so on. Every line has its C_1 and every lineoid its C_2 of any given complex. A flat axal pencil will be said to belong to a given hyperpencil when the latter contains the planes of the former. The *degree* of a C_1 or a C_2 will signify the number of planes common to the cone and an arbitrary flat axal pencil belonging to the hyperpencil to which the cone belongs. It should be noted that as the notions, *locus* and *envelope,* of the Plücker geometry correspond respectively to *envelope* and *locus* in the present theory, so also the notions of *order* and *class* in the former doctrine correspond to those of *class* and *order* in the latter. Thus the curve of a line complex is an *envelope* of *lines,* but its correlate, C_1 of a plane complex, is a *locus* of *planes.* The degree of C_1 will be called the *order* of this cone, and the degree of C_2 will be called its *class.* We have immediately the proposition : *The degree of a complex is equal to the order of any of its C_1's and to the class of any of its C_2's.*

16. A complex of first degree is said to be *linear.* A C_1 of such a complex is of order 1, it is a flat axal pencil, to be viewed as a lineoid of collinear planes; while a C_2 of the linear complex, being of class 1, is also a flat axal pencil, to be viewed, however, as a line of collineoidal planes: the lineoid C_1 is a *locus* of the planes of the flat axal pencil; the line C_2 is an *envelope* of the planes of the flat axal pencil: the line and the lineoid are thus but reciprocal phases of one con-

figuration, just as in line geometry the flat pencil is regarded now as a point and again as a plane. Given an arbitrary linear complex of planes. Of these there pass through any line whatever a single infinity of planes all contained in a lineoid and constituting a flat axal pencil; the lineoid so determined will be called the *polar* lineoid of the given line. Reciprocally every lineoid contains a single infinity of the planes of the given complex and these, too, are collinear, constituting a flat axal pencil; the axis, or line so determined, will be called the *polar* line of the given lineoid. Accordingly *with respect to any linear plane complex, every lineoid has a polar line, and every line has a polar lineoid.* Every lineoid or line is *united in position* with its polar line or lineoid.

17. These propositions, showing the distribution of the planes of a linear complex, are of such fundamental importance as to justify their separate establishment by analytical means. As a preliminary we will show that a linear plane complex is representable by an equation of first degree in ν_i, and conversely, that every such equation defines such a complex.

Let the equation

$$F(\nu_i) = 0$$

represent a linear complex of planes. The identity

$$\Omega(\nu) = 0$$

is, of course, supposed given. Denote by ν_i' and ν_i'' any two collineoidal planes. We have seen that the coordinates of the planes of the flat axal pencil determined by ν_i' and ν_i'' are

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i''.$$

The condition that one of these planes shall belong to F is

$$F(\lambda_1 \nu_i' + \lambda_2 \nu_i'') = 0.$$

Since by definition of F, only one plane of the pencil belongs to F, the last equation must be linear in $\lambda_1 : \lambda_2$, and is, therefore, of the form

$$\Sigma c_i \nu_i = 0.$$

The converse is obviously correct.

Now let π denote any plane whatever and let (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4)

be any two generating lineoids of π. Then for coordinates of π we may take

$$\begin{cases} v_1 = x_1 y_2 - x_2 y_1, & v_4 = x_2 y_3 - x_3 y_2, \\ v_2 = x_1 y_3 - x_3 y_1, & v_5 = x_2 y_4 - x_4 y_2, \\ v_3 = x_1 y_4 - x_4 y_1, & v_6 = x_3 y_4 - x_4 y_3. \end{cases}$$

The condition that π shall belong to the complex

$$\Sigma c_i v_i = 0,$$

may, therefore, be written

$$\begin{aligned}(c_1 y_2 + c_2 y_3 + c_3 y_4) x_1 + (-c_1 y_1 + c_4 y_3 + c_5 y_4) x_2 \\ + (-c_2 y_1 - c_4 y_2 + c_6 y_4) x_3 + (-c_3 y_1 - c_5 y_2 - c_6 y_3) x_4 = 0.\end{aligned}$$

This equation, if the y's be regarded as fixed and the x's as variable represents a straight line, and as the equation is satisfied by $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, $x_4 = y_4$, this line lies in the lineoid y. It thus appears that the *planes π of a given lineoid y that belong to a given linear complex envelope a line, the polar of a given lineoid. They constitute a flat axal pencil within y.*

In like manner, if π be supposed given by two of its lines ξ and η, reasoning analogous to the foregoing will show that the *planes of a given line y that belong to a given linear complex have for locus a lineoid, polar of the given line.*

The flat axal pencils which are thus determined, one for each line and one for each lineoid, by any given linear complex, may be called the *pencils of the complex.*

Denote by l any line and by L any lineoid containing l, and consider L' and l', the polars respectively of l and L with respect to a given linear plane complex C. The plane (L, L') being contained in L' and containing the polar l of L', belongs to C, and, therefore, as it is contained in L, it contains l'. Consequently, l' lies in L'. Hence, *the polars of a line and lineoid united in position are themselves united in position.*

Let π be any plane. Every generating lineoid L of π is united in position with every generating line l of π. Hence, every polar line l' of the L's is united in position with every polar lineoid L' of the l's Hence, the L''s and the l''s generate one and the same plane π'. Two planes π and π' thus related will be called *conjugate planes with respect to the given complex. Two conjugate planes are such that either of them is the locus (or envelope) of the polar lines (or lineoids) of the generating lineoids (or lines) of the other.*

18. It thus appears that a linear complex of planes serves as a dualistic transformation establishing a unique and reciprocal correspondence* between lines and lineoids, and between planes and planes. In this correspondence each plane of the complex corresponds to itself; for obviously, if π belongs to the complex, π and its conjugate π' coincide, i. e., *every plane of the given complex is self-conjugate or self-polar with respect to that complex.* On the other hand, no other plane is self-conjugate. In fact, if two conjugates π and π' are not planes of the complex, they are not collineoidal, for suppose them contained in a lineoid L; the polar line of L lies in both π and π', and hence these planes belong to the complex and consequently coincide. Therefore, *two conjugate planes coincide and so belong to the complex or else they are non-collinear and so do not belong to the complex.* This proposition is a corollary to the following: *If two conjugates, π_1 and π_1', are each collineoidal with the plane π, the latter belongs to the complex.* To prove this proposition, denote by L_1 the lineoid determined by π_1 and π, and by L_1' that determined by π_1' and π; the polar line l_1 of L_1 lies in π_1', and hence in L_1', and the polar line l_1' of L_1' lies in π_1 and hence in L_1; therefore, l_1 and l_1' are both lines of π, the common plane of L_1 and L_1'; hence π belongs to the complex.

Let π, any plane of the complex, be collineoidal with a plane π_1. If L be the lineoid containing π and π_1, the polar line l of L lies in π, and as L contains π_1, l also lies in π_1', the conjugate of π_1'; hence, *if a plane of a complex is collineoidal with any other plane, it is also collineoidal with the conjugate of the latter.*

19. The foregoing and additional properties of conjugate planes may be investigated analytically as follows: The condition

$$\Omega(\nu', \nu) = 0 = \Sigma \frac{\partial \Omega(\nu')}{\partial \nu_i'} \nu_i$$

that the planes ν_i' and ν_i shall be collineoidal (or collinear) will assume the form

$$\nu_4'\nu_1 + \nu_5'\nu_2 + \nu_6'\nu_3 + \nu_1'\nu_4 + \nu_2'\nu_5 + \nu_3'\nu_6 = 0$$

on taking $\Omega(\nu)$ to be of the form

$$\nu_1\nu_4 + \nu_2\nu_5 + \nu_3\nu_6 .$$

* Exceptions to the one-to one character of this correspondence will be noted at a later stage.

The condition, being linear in ν_i' and ν_i, shows that the assemblage of planes of which each is collineoidal with a given plane is a linear complex. Such a linear complex will be called a *special* complex. The condition that the complex

$$\Sigma c_i \nu_i = 0$$

shall be a special complex is thus seen to be

$$\Omega(c) = c_1 c_4 + c_2 c_5 + c_3 c_6 = 0,$$

which is identical with the condition in the Plücker geometry that every line of a line complex shall have a point in common with a given line. Employing Klein's terminology for the line theory, we will call the quadratic form $\Omega(c)$ the *invariant* of the complex. In case of a special complex the plane which is collineoidal with each plane of the complex will be called the *director* plane or *directrix* of the complex, this plane being the analogue of the directrix of the special line complex of the line theory.

Let
$$\Sigma c_i x_i = 0 \tag{1}$$

be an arbitrary chosen complex, and denote by ν_i' any given plane. The latter is director plane of the special complex

$$\Omega(\nu', \nu) = 0. \tag{2}$$

The planes common to (1) and (2) are identical with the planes of the flat axal pencils (lines) that are polar to the generating lineoids of ν_i' with respect to (1). Denote by ν_i'' the plane common to *any* two of these pencils. The planes common to (1) and the special complex

$$\Omega(\nu'', \nu) = 0 \tag{3}$$

are identical with the planes of the polar flat axal pencils (lines) of the generating lineoids of ν_i''. The plane ν_i''' common to any two of these pencils is identical with ν_i'; for if L_1 and L_2 be the lineoids whose polar lines (pencils) give ν_i'', then, as the polar line l of any lineoid L of ν_i'' must lie in both L_1 and L_2, it follows that ν_i' is the locus of such polar lines l. We have accordingly the proposition: *Any pair of complexes C and C', of which one of them as C' is special, determines a third special complex C'' such that the assemblage of planes common to C and C' is identical with the assemblage common to C and C''.*

The director planes of C' and C'' are evidently *conjugates* with respect to C. In order, therefore, that ν_i' and ν_i'' shall be conjugate planes with respect to (1),

it is necessary and sufficient that

$$\Sigma c_i \nu_i = \lambda_1 \Omega(\nu', \nu) + \lambda_2 \Omega(\nu'', \nu), \tag{4}$$

whence

$$\left.\begin{aligned} c_1 &= \lambda_1 \nu_4' + \lambda_2 \nu_4'', & c_4 &= \lambda_1 \nu_1' + \lambda_2 \nu_1'', \\ c_2 &= \lambda_1 \nu_5' + \lambda_2 \nu_5'', & c_5 &= \lambda_1 \nu_2' + \lambda_2 \nu_2'', \\ c_3 &= \lambda_1 \nu_6' + \lambda_2 \nu_6'', & c_6 &= \lambda_1 \nu_3' + \lambda_2 \nu_3'', \end{aligned}\right\} \tag{5}$$

for some value of the ratio $\lambda_1 : \lambda_2$; or, writing the complexes in the form

$$\Sigma \frac{\partial \Omega(c)}{\partial c_i} \cdot \frac{\partial \Omega(\nu)}{\partial \nu_i} = 0, \quad \Sigma \frac{\partial \Omega(\nu)}{\partial \nu_i} \nu_i' = 0, \quad \Sigma \frac{\partial \Omega(\nu)}{\partial \nu_i} \nu_i'' = 0, \tag{6}$$

the condition may be written

$$\frac{\partial \Omega(c)}{\partial c_i} = \lambda_1 \nu_i' + \lambda_2 \nu_i'', \qquad (i = 1, 2, \ldots, 6). \tag{7}$$

By the aid of the condition

$$\Omega\left(\frac{\partial \Omega(c)}{\partial c} - \lambda_1 \nu'\right) = 0 \tag{8}$$

that the ν_i'' shall be coordinates of a plane, we readily find

$$\lambda_1 = \Omega(c) : \Sigma c_i \nu_i', \tag{9}$$

whence the coordinates of the conjugate ν'' of ν_i' are given by

$$\lambda_2 \nu_i'' = \frac{\partial \Omega(c)}{\partial c_i} - \frac{\Omega(c)}{\Sigma c_i \nu_i'} \nu_i'. \tag{10}$$

Four cases may arise:

$$\Omega(c) \neq 0, \quad \Sigma c_i \nu_i' \neq 0, \tag{a}$$
$$\Omega(c) \neq 0, \quad \Sigma c_i \nu_i' = 0, \tag{b}$$
$$\Omega(c) = 0, \quad \Sigma c_i \nu_i' \neq 0, \tag{c}$$
$$\Omega(c) = 0, \quad \Sigma c_i \nu_i' = 0. \tag{d}$$

In (a), which is the general case, the complex (1) is non-special and the plane ν_i' does not belong to (1). From the symmetry of (7), in respect to ν_i' and ν_i'', it appears that ν_i'' does not belong to (1). Hence, *of two conjugates with respect to a complex either both belong or neither belongs to the complex.* Equation (10) shows that under (a) to two planes ν_i' there correspond two planes ν_i'' and reciprocally (cf. §18). The equation

$$\Omega(\nu', \nu'') = 0 \tag{11}$$

signifies indifferently that ν_i'' belongs to (2) or that ν_i' belongs to (3); the director plane of a special complex may be considered as belonging to that complex; hence, if (11) be true, both ν_i' and ν_i'' belong to both (2) and (3) and hence also to (1), but this is contrary to (a). Hence, *two conjugates that do not belong to the given complex are non-collineoidal.*

In (b), ν_i' belongs to (1), which is non-special; $\lambda_1 = \infty$, and we have from (7)

$$\lambda_1 : \lambda_2 = -\nu_i'' : \nu_i',$$

which shows that every plane of (1) is self-conjugate.

In (c) the complex (1) is special, $\lambda_1 = 0$, and

$$\lambda_2 \nu_i'' = \frac{\partial \Omega(c)}{\partial c_i};$$

hence *the conjugate of any ν_i' with respect to a special complex not containing ν_i' is the director plane of the complex.*

In case (d), λ_1 is indeterminate; the meaning is that the conjugate of any ν_i' with respect to a special complex containing ν_i' is indeterminate, i. e., may be indifferently taken to be either ν_i' itself, as in (b), or the director plane, as in (c). One and the same line l is polar to all the lineoids of a given ν_i, and l is the intersection of ν_i and the director plane.

The results under the four cases may be summarized thus: *Given a complex C and let π stand for plane; π is self-conjugate or not so according as it belongs or does not belong to C; if π_1 and π_2 be any two planes, their conjugates are distinct or not according as C is non-special or special. In case of C special, the director plane is conjugate with all planes, itself included.*

IV.—*Linear Congruences of Planes, and Pencils of Complexes.*

20. A two-parameter system of planes, i. e., a system in which a plane has two degrees of freedom, has been named *congruence* (§11). It follows that the assemblage of planes that are common to two complexes is a congruence. The two-dimensional assemblage of planes constituting an axal pencil is a congruence. In like manner, the manifold of planes contained in a lineoid constitute a congruence, i. e., a bundle of planes is a congruence. A congruence being given, the number of planes it has in common with an arbitrary axal pencil will be called its *order*, while the number it has in common with an arbitrary bundle will be

called its *class.* In other words, the terms order and class of a congruence signify respectively the number of planes common to the congruence and an arbitrary line and the number common to the congruence and an arbitrary lineoid. The notions *order* and *class* of a plane congruence are seen to correspond respectively to *class* and *order* of a line congruence in the Plücker theory, the order of a line congruence being the number of its lines that go through an arbitrary point, and its class the number that lie in an arbitrary plane. A bundle of planes is of class one and order zero, while an axal pencil is of class zero and order one, just as in the Plücker theory a plane of lines is of class one and order zero, while a sheaf of lines is of order one and class zero.

21. We shall be chiefly concerned in this chapter with such congruences as are definable by two linear complexes of planes. Such congruences may be themselves called *linear.* We have seen that, given a linear complex, an arbitrary lineoid contains a flat axal pencil of planes belonging to the complex. It follows that an arbitrary lineoid contains one and but one plane of a given linear congruence. Reciprocally, an arbitrary line is contained in one and but one plane of the given congruence. It thus appears that a congruence composed of the planes common to two linear complexes is of order one and class one.

The assemblage of complexes represented by the equation

$$\lambda\Sigma c_i\nu_i + \lambda'\Sigma c_i'\nu_i \equiv \Sigma\,(\lambda c_i + \lambda' c_i')\,\nu_i = 0, \tag{1}$$

the c's being supposed given, and the λ's being parameters, will be called a *pencil* of complexes. The given complexes c_i and c_i', which determines it, may be called the *fundamental* complexes of the pencil.

If $\lambda_1 : \lambda_1'$ and $\lambda_2 : \lambda_2'$ be any two complexes of the pencil (1), this last is identical with the pencil

$$\Sigma\,[(\sigma\lambda_1 + \sigma'\lambda_2)\,c_i + (\sigma\lambda_1' + \sigma'\lambda_2')\,c_i']\,\nu_i = 0,$$

the σ's being parameters, for, in order to identify any given complex of either pencil with one of the other, we need only the relation

$$\lambda_1 : \lambda_2' = \sigma\lambda_1 + \sigma'\lambda_2 : \sigma\lambda_1' + \sigma'\lambda_2',$$

which, it is plain, can always be found. Hence, *the pencil of complexes determined by any two complexes of a given pencil is identical with the given pencil.*

22. It is obvious that the congruence defined by any two complexes c_i and c_i' is common to all the complexes of the pencil determined by c_i and c_i', and from the foregoing theorem it follows that the congruence may be regarded at will as defined by any pair whatever of the complexes of the pencil. A pencil of complexes and the congruence determined by a pair of the complexes may be said to *correspond.* In this way, to every pencil there corresponds a congruence, and conversely.

The condition that a complex of the pencil determined by the complexes c_i and c_i' shall be *special* is

$$\Omega(\lambda c + \lambda' c') \equiv \Omega(c)\lambda^2 + \Omega(c')\lambda'^2 + 2\Omega(c, c')\lambda\lambda' = 0. \tag{3}$$

Two cases are to be considered according as the roots of this equation, regarded as an equation in $\lambda : \lambda'$, are

(a) determinate

or (b) indeterminate.

Under (a), the general case, there fall two sub-cases:

(a′) $\Delta(c, c') \neq 0$,

(a″) $\Delta(c, c') = 0$,

where $$\Delta(c, c') \equiv \Omega(c)\,\Omega(c') - \Omega^2(c, c').$$

In (a) equation (3) has two and but two roots. These, which may be real or conjugate imaginaries, are definite and distinct. The pencil accordingly contains *two definite and distinct special complexes.* The congruence corresponding to the pencil consists of the planes that are at the same time collineoidal with the two director planes of the special complexes. These director planes may therefore be called the director planes of the congruence.

Denote by ν_i' and ν_i'' the director planes in question. The special complexes of the pencil

$$\Sigma(\lambda c_i + \lambda' c_i')\nu_i = 0 \tag{4}$$

will be given by the equations

$$\Omega(\nu', \nu) = 0, \quad \Omega(\nu'', \nu) = 0. \tag{5}$$

As the director plane of a special complex is, with respect to that complex, conjugate to all planes whatever, the planes ν_i' and ν_i'' are conjugate with respect

to both complexes (5). These two planes are conjugate (§19, Eq. 7) with respect to all and only the complexes of the pencil

$$\Sigma\,(\rho w_i' + \rho' w_i'')\,\nu_i = 0 \tag{6}$$

where

$$\begin{cases} w_1',\, w_2',\, w_3',\, w_4',\, w_5',\, w_6' = \nu_4',\, \nu_5',\, \nu_6',\, \nu_1',\, \nu_2',\, \nu_3', \\ w_1'',\, w_2'',\, \ldots\ldots\ldots\ldots = \nu_4'',\, \nu_5'',\, \ldots\ldots\ldots\ldots \end{cases}$$

Therefore (6) contains the complexes (5), and as these are also contained in (4), it follows that (4) and (6) are identical. Hence, *the two director planes of a linear congruence are conjugate with respect to every complex of the corresponding pencil.* Also *if two planes are conjugate with respect to each of two complexes, they are the director planes of the common congruence of those complexes.*

In sub-case (a) the two roots of (3) are equal, the two director planes *coincide,* are one plane. This plane is to be counted twice, once as belonging to the one and once as belonging to the other, of the two (coincident) special complexes. This double plane ($\nu_i' \equiv \nu_i''$) is, therefore, a plane of the congruence. All planes of the congruence are collineoidal with ν_i', but the converse is not true, for while a plane of a congruence has two, a plane merely required to be collineoidal with a given plane has three, degrees of freedom. The precise way in which a plane having three degrees of freedom as specified, loses one degree on being required to belong to a linear congruence having the given plane ν_i' for double directrix, may be seen as follows: If π belongs to the congruence, the lineoid (π, ν_i'), since it contains two planes of each complex of the pencil, is, in regard to each complex, the polar lineoid of the line (π, ν_i'). Of the ∞^2 planes containing this line, only ∞^1 planes, viz., those contained in the lineoid (π, ν_i'), belong to the congruence. If, therefore, a bilinear relation be set up between the generating lines and the generating lineoids of any plane ν_i', then the assemblage of planes obtained by taking all and only flat axal pencils that are contained in the lineoids corresponding to the axes (lines) is a linear congruence having ν_i' as double directrix.

The necessary and sufficient conditions for the occurrence of case (b) are

$$\Omega\,(c) = 0, \quad \Omega\,(c') = 0, \quad \Omega\,(c, c') = 0.$$

The first two of these equations indicate that the fundamental complexes c_i and c_i' are special, and the third equation signifies that the director planes of c_i and c_i' are collineoidal. The indetermination of $\lambda : \lambda'$ shows that the pencil is com-

posed of special complexes. The corresponding congruence has for director planes *any* pair of an infinity of planes, the director planes of the pencil, all of which belong to the congruence. We may readily prove that *these director planes constitute a flat axal pencil.* For if

$$\Sigma(\lambda c_i + \lambda' c') \nu_i = 0$$

be any complex of the pencil of complexes and if ν_i' be the corresponding director plane, then

$$\nu_i' = \frac{\partial \Omega(\lambda' c + \lambda' c')}{\partial(\lambda c_i + \lambda' c_i')}, \qquad (i = 1, \ldots, 6)$$

or

$$\nu_i' = \lambda \frac{\partial \Omega(c)}{\partial c_i} + \lambda' \frac{\partial \Omega(c')}{\partial c_i'},$$

which shows that as $\lambda : \lambda'$ varies through all real values, the director plane ν_i' generates a flat axal pencil p, namely, that determined by the director planes of the complexes c_i and c_i'. The congruence consists of all the planes of which each is collineoidal with each plane of p. It is accordingly composed of two hyperpencils, one of each kind: the *bundle* contained in the lineoid containing the planes of p, and the *axal pencil* having for its axis the axis of p. Each of these components is of itself a congruence, the former of order zero and class one, the latter of order one and class zero, showing what should be the case that the congruence under consideration is of order one and class one.

23. The propositions of line geometry that correspond to the foregoing may be briefly stated as follows:

Case (a), sub-case (a′). The linear congruence has two distinct non-intersecting directrices (lines). The assemblage of lines joining the latter is the congruence. The directrices are conjugate lines with respect to every line complex of the pencil of complexes defining the congruence.

Case (a), sub-case (a″). The congruence has one (double) directrix. If a bilinear relation be established between the points and the planes of any given line l, this line is the double directrix of the congruence obtained by taking all and only the lines of such flat pencils as lie each in the plane that corresponds to the vertex (a point of l) of the pencil.

Case (b). The congruence has an infinity of directrices, which belong to it and which constitute a flat pencil of lines. The congruence is composed of two congruences: the assemblage of lines of the plane of the pencil and the sheaf

of lines containing the vertex of the pencil, the former component being of order zero and class one, while the latter is of order one and class zero.

24. The geometric properties of the discriminant Δ of $\Omega(\lambda c + \lambda' c')$ show that Δ is an invariant both under a reversible linear transformation of the variables v and under such a transformation of the λ's; for the coincidence or non-coincidence of the director planes of a congruence is independent alike of the coordinates employed and of the particular choice of a pair out of the corresponding pencil of complexes for fundamental complexes. We will merely state without proof that, if M_1 be the modulus of the v-transformation and Δ_1 be the new discriminant, and if M_2 be the modulus of the λ-transformation and Δ_2 the resulting discriminant, then

$$\begin{cases} \Delta_1 = M_1^4 \Delta, \\ \Delta_2 = M_2^2 \Delta. \end{cases}$$

The λ-transformation being merely equivalent to replacing the fundamental complexes c_i and c_i' by a new pair as $\lambda_1 c_i + \lambda_1' c_i'$ and $\lambda_2 c_i + \lambda_2' c_i'$, we may write

$$\Delta_2 = \Delta(\lambda_1 c + \lambda_1' c', \lambda_2 c + \lambda_2' c') = (\lambda_1 \lambda_2' - \lambda_2 \lambda_1')^2 . \Delta.$$

As the vanishing or non-vanishing of Δ is not a mark of any particular pair of complexes of the pencil but characterizes the pencil as such, we may name $\Delta(c_1 c')$ the *discriminant of the pencil* $\lambda c_i + \lambda' c_i'$ *or of the corresponding congruence.* A pencil of complexes being given, it contains two distinct or two coincident special complexes, and the corresponding congruence has two distinct or two coincident director planes, according as the discriminant vanishes or does not vanish.

V.—*Projective Transformations by Means of Complexes. Orthogonal Complexes: Involution.*

25. We have seen that a linear complex of planes serves as a means of dualistic transformation according to which lines or lineoids correspond to lineoids or lines, and planes correspond to planes. If c_i be any linear complex and if π and π' be any pair of conjugate planes with respect to c_i, then the generating lines or lineoids of π or π' are projectively related through c_i to the generating lineoids or lines of π' or π, any line or lineoid being with respect to c_i the polar of the corresponding lineoid or line. If l_1, l_2, l_3, l_4 be any four lines of π or π',

and L_1, L_2, L_3, L_4 be the corresponding lineoids (of π' or π), the anharmonic ratio of the l's is equal to that of the L's, i. e.,

$$(l_1 l_2 l_3 l_4) = (L_1 L_2 L_3 L_4).$$

If π belong to c_i, π is self-conjugate, and if generated by l it will at the same time be generated by L, the (polar) correspondent of l. The plane being at once a locus of lines (a pencil of lines) and an envelope of lineoids (a pencil of lineoids), we have the proposition: *Any linear complex of planes establishes a projective correspondence between the lines and the lineoids of each of its planes.*

If now we suppose π to be common to the two complexes c_i and c_i', and if to the lineoids L_1, L_2, L_3, L_4 of π there correspond with reference to c_i, the lines l_1, l_2, l_3, l_4 and with reference to c_i' the lines l_1', l_2', l_3', l_4', then, as

$$(L_1 L_2 L_3 L_4) = (l_1 l_2 l_3 l_4),$$
$$(L_1 L_2 L_3 L_4) = (l_1' l_2' l_3' l_4'),$$

we have
$$(l_1 l_2 l_3 l_4) = (l_1' l_2' l_3' l_4').$$

Accordingly, if we regard π as *two superposed* pencils, viz., of lines l (associated with c_i) and of (the same) lines l' (associated with c_i'), it is seen that these pencils are brought into projective relation by means of the complexes c_i and c_i'. Reciprocally, π may be conceived as two superposed pencils of (its generating) lineoids, L and L', and these, too, are projectively related through the complexes in question. Given either of the line (or lineoid) pencils, the other pencil (of the same kind) is obtainable from the given one by means of a linear line (or lineoid) transformation of π. Hence, *the assemblage of two given complexes plays the rôle of a definite linear transformation at the same time of the lines and the lineoids of any given plane of the corresponding congruence.*

As π is common to all the complexes of the pencil

$$\Sigma(\lambda c_i + \lambda' c_i') v_i = 0, \tag{7}$$

it follows that the lines and lineoids of π are transformed by every pair, $\lambda' : \lambda = k_1$, $\lambda' : \lambda = k_2$, of these complexes. The equation of the transformation will assume its simplest form when referred to its *foci*. What then are these? Denote by π_d and π_d' the director planes of the congruence, and let l_d and l_d' be the lines and L_d and L_d' the lineoids determined by π and the director planes. As π_d and π_d' are conjugates with respect to both k_1 and k_2, l_d and L_d' are each the other's polar in regard to both k's. Similarly, l_d' and L_d. Hence l_d and l_d' are the foci

of the line transformation, and L_a and L'_a are the foci of the lineoid transformation, effected on π by the pair k_1, k_2 of complexes. Taking l_a and l'_a as base elements of homogeneous coordinates, $z = x_1 : x_2$, of the lines of the pencil determined by l_a and l'_a, the line transformation assumes the form

$$z' = \rho z.$$

In like manner, if z be interpreted as coordinates of the lineoids of the pencil lineoids referred to L_a and L'_a as base elements, the lineoid transformation takes the form

$$z' = \rho' z.$$

26. In case of the line transformation, the *characteristic* constant ρ is the anharmonic ratio formed by any pair z and z' of corresponding lines of π with the foci l_a and l'_a. Writing

$$\alpha = \frac{1}{2i} \log \rho,$$

α will be the (generalized) angle of the lines z and z'. If the foci be the isotropic lines of the pencil in question, α will be the ordinary angle of z and z'. In like manner α', where

$$\alpha' = \frac{1}{2i} \log \rho'$$

is the angle between any pair z and z' of corresponding lineoids.

42. Since the transformation is determined completely by the given complexes k_1 and k_2, it must be possible to express ρ and ρ' in terms of the parameters k_1, k_2 and the coefficients of the fundamental complexes c_i and c'_i. We proceed to determine such expressions for ρ.

Let L be any lineoid of π. With respect to each complex of the pencil (7), L has a polar line l. By virtue of this one-to-one correspondence, if k_1, k'_1, k_2, k'_2 be any four of the complexes and l_1, l_2, l_3, l_4 the corresponding polar lines of L, we shall have

$$(l_1 l'_1 l_2 l'_2) = (k_1 k'_1 k_2 k'_2).$$

If, now, we replace k'_1 and k'_2 respectively by s_1 and s_2, where $\lambda' : \lambda = s_1$, $\lambda' : \lambda = s_2$ are the *special* complexes of (7), l'_1 and l'_2 will be replaced by the foci l_a and l'_a and we shall obtain, since l_1 and l_2 are any pair of correspondents,

$$\rho = (l_1 l_a l_2 l'_a) = (K_1 S_1 K_2 S_2).$$

In precisely like manner, it may be shown that

$$\rho' = (k_1 s_1 k_2 s_2),$$

whence

$$\rho = \rho', \quad \alpha = \alpha'.$$

Hence, *The line and lineoid transformation effected by two given complexes k_1, k_2 on any plane π of their common congruence are algebraically identical. The transformation remains unchanged if π be supposed to generate the congruence. The angle of two corresponding lines of a given π is constant. The same is true of the angle of two corresponding lineoids.*

27. The angle of any pair of lines of one plane is equal to that of any pair of any other plane of the congruence. The two pairs are, however, not congruent (in the ordinary sense). The equality is merely arithmetic, not geometric. The systems of measurement in the one plane and in the other are not the same. The foci (the Cayleyan *absolute*) in the one plane are not congruent with the foci in case of the other. Like remarks apply to the lineoid transformations, and to the equation $\alpha = \alpha'$.

28. There is a simple infinity, ∞^1, of linear transformations having the same assigned foci. On the other hand, there are ∞^2 pairs of complexes defining one and the same congruence and each pair gives a line (lineoid) transformation of the planes. It is, therefore, to be expected that, if one pair gives a transformation, there are ∞^1 pairs giving the same transformation. That such is the case appears in the equation

$$\rho = (k_1 s_1 k_2 s_2),$$

which, being linear in k_1 and in k_2, is satisfied by ∞^1 of pairs of values of the k's This statement is independent of the value of ρ, and as the foci in case of any plane are the same for all pairs of k's, we have the proposition: *The lines (lineoids) of every plane of a congruence are transformed by ∞^1 distinct transformations by pairs of complexes of the corresponding pencil of complexes, each transformation is effected by ∞^1 pairs, and all transformations of any given plane have the same foci*

Holding ρ fixed, the above equation may be regarded as a *linear transformation of the pencil* (7) *of complexes.* The foci of the transformation are the special complexes s_1 and s_2, for on writing

$$\rho = \frac{(k_1 - s_1)(k_2 - s_2)}{(s_1 - k_2)(s_2 - k_1)},$$

it is seen that k_1 and k_2 coincide when and only when one of them coincides with s_1 or s_2 (the s's are here supposed distinct). The constant ρ is, then, the anharmonic ratio formed by *any* pair k_1 and k_2 of complexes with the pair of special complexes s_1 and s_2. We may therefore call

$$\alpha \equiv \frac{1}{2i} \log \rho,$$

the *angle of the complexes k_1 and k_2*. Hence the proposition: *The angle of any pair of complexes is equal to the angle between any two corresponding lines (lineoids) in the transformation effected by the given complexes upon any plane of their common congruence.*

The case where $\rho = -1$ and $\alpha = \frac{\pi}{2}$ is of special interest. In this case the complexes k_1 ank k_2 may be said to be *orthogonal*, or, since the pair k_1, k_2 is *harmonic* to the pair s_1, s_2, the complexes k_1 and k_2 may be said to be *in involution* with respect to the two special complexes. The condition for such orthogonality or involution is

$$2k_1 k_2 - (s_1 + s_2)(k_1 + k_2) + 2s_1 s_2 = 0, \tag{8}$$

which shows that *every complex of a pencil of complexes is in involution with one and (in general) only one other complex of the pencil.*

It is plain also that *any pair of corresponding lines (lineoids) of a plane transformed by two orthogonal complexes is harmonic to the pair of foci in that plane.*

If, in equation (8), we let $k_1 = 0$, then $k_2 = 2s_1 s_2 : (s_1 + s_2)$, and this will be ∞ when and only when $s_1 + s_2 = 0$, or, since the s's are the roots of equation (3), the condition is

$$\Omega(c, c') = 0.$$

The values $k_1 = 0$, $k_2 = \infty$ correspond to the fundamental complexes c_i and c_i' of the pencil (7). The vanishing of $\Omega(c, c')$ is, therefore, the condition that the complexes c_i and c_i' shall be orthogonal. This function of the coefficients c is that which, in the Plücker geometry, Klein has called the *simultaneous invariant* of the complexes c_i and c_i'.

29. Suppose that the plane ν_i' belongs to c_i', and that ν_i'' is the conjugate of ν_i' with respect to c_i, then (IV, Eq. 7),

$$\frac{\partial \Omega(c)}{\partial c_i} = \lambda_1 \nu_i' + \lambda_2 \nu_i'',$$

whence
$$\Sigma \frac{\partial \Omega(c)}{\partial c_i} c'_i = \lambda_1 \Sigma c'_i \nu'_i + \lambda_2 \Sigma c'_i \nu''_i,$$
i. e.,
$$2\Omega(c, c') = \lambda_2 \Sigma c'_i \nu''_i.$$
Since, by hypothesis,
$$\Sigma c'_i \nu'_i = 0.$$

Accordingly, if ν''_i belongs to c'_i, the simultaneous invariant vanishes. Therefore, *if two complexes are such that one of them contains a plane conjugate to one of its planes with respect to the other complex, the complexes are orthogonal.*

It is, moreover, seen that, if ν''_i is in c'_i, then ν''''_i the conjugate of any other plane ν'''_i of c'_i is also in c'_i, i. e., the planes of c'_i fall into pairs of conjugates with respect to c_i, and, as the relation is a reciprocal one, the planes of c_i fall into pairs of conjugates with respect to c'_i. Two complexes thus related may be said to be each its own conjugate, or *self-conjugate*, or self-polar, with respect to the other. *This property of self-conjugateness characterizes both complexes of every orthogonal pair and might be taken as the defining property of orthogonality.*

30. The notion of orthogonality has as yet been attached only to pairs of non-special complexes. Nevertheless if (following Klein and Koenigs in the line theory) we agree to say that two complexes are orthogonal *always* when their simultaneous invariant is zero, then it readily follows: (1) *That every special complex is orthogonal to all complexes containing its director plane, and conversely;* (2) *that the necessary and sufficient condition for the orthogonality of two special complexes is that their director planes be collineoidal.*

Consideration of the *net*
$$\lambda_0 \Sigma c_i \nu_i + \lambda_1 \Sigma c'_i \nu_i + \lambda_2 \Sigma c''_i \nu_i = 0,$$
and the *web*
$$\lambda_0 \Sigma c_i \nu_i + \dots\dots + \lambda_3 \Sigma c'''_i \nu_i = 0,$$
of complexes, which correspond respectively to the one-parameter system (configuration, plane-series) and the zero-parameter system of planes, is reserved.

COLUMBIA UNIVERSITY, *Aug.* 10, 1901.

On the Functions Representing Distances and Analogous Functions.

By H. F. Blichfeldt.

Introduction.

Having given the analytical expression for the distance between two points x_1, y_1; x_2, y_2 in the plane, it is a simple matter to prove that the six distances connecting any four points in the plane are connected by one relation, and to find this relation.*

Again, having given the analytical expression for the angle between two planes in space, $a_1 x + b_1 y + z = c_1$, $a_2 x + b_2 y + z = c_2$, we easily find that one relation exists between the six angles made by any four planes in space.

In these examples certain functions of the analytical expressions representing a distance or an angle, namely

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 \quad \text{and} \quad \frac{(a_1 a_2 + b_1 b_2 + 1)^2}{(a_1^2 + b_1^2 + 1)(a_2^2 + b_2^2 + 1)},$$

have this in common, that each is a function of four independent variables occurring in pairs, as x_1, y_1; x_2, y_2, in such a way that the interchange of the constituents of one pair with the corresponding constituents of the other pair leaves the functions unaltered. Moreover, the six functions representing the six distances between four points or the six angles made by four planes are connected by one relation, which, however, is fundamentally different in the two cases.

Other geometric magnitudes might be chosen whose analytical expressions would possess the same characteristic properties, and the question then arises,

* See for example Salmon's "Higher Algebra," 4th Edition, p. 27.

what are the typical forms of all such expressions? This question, slightly extended, will be answered in the following paper, and a few geometrical applications of the solution will be given (§§4–7). Using the coordinates of points of the plane to represent the pairs of variables in the functions sought, we shall formulate this question as follows:

Calling a function of the coordinates of two points of the plane, left unaltered by interchanging the two points, a two-point function, it is desired to find the two-point functions possessing the following property:

The ten two-point functions of the ten pairs of points of any five points are connected by two or more relations independent of the coordinates of the points.

Some of the more general terms and properties of Lie's Continuous Groups, such as *infinitesimal transformations, commutators* ("Klammerausdrücke"), *point-invariants, similarity of groups,* will be used and will be supposed known. The phrase "change of variables," or "transformation of the variables," frequently employed in the following analysis, is understood to mean a change in the coordinate system of the plane of reference. It, therefore, denotes a transformation of the form

$$x_i = \phi(x_i', y_i'), \quad y_i = \psi(x_i', y_i'),$$

ϕ and ψ being the same functions whatever may be the subscript i. We shall, for brevity, write "*the function* (i, j)" in place of "*the two-point function of the points* $x_i, y_i;\ x_j, y_j$."

In order to determine the two-point functions having the properties stated above, it is convenient to divide them into three classes, according as the ten two-point functions of five points are connected by just two relations, by three, or by more than three such.

DETERMINATION OF THE TWO-POINT FUNCTIONS.

I.—*The Ten Two-Point Functions of Five Points are Connected by Just Two Relations.*

§1.

Let the five points be $x_1, y_1;\ x_2, y_2;\ \ldots\ ;\ x_5, y_5$, and the two-point functions $f(x_i, y_i;\ x_j, y_j)$; $i, j = 1, 2, 3, 4$, or 5, $i \neq j$. Only eight of these are independent, according to hypothesis. We can therefore construct, in the usual way, two

independent linear partial differential equations satisfied by all the ten two-point functions. Denoting the partial differential coefficients of any solution f of these equations with respect to x_i and y_i by p_i and q_i respectively, the differential equations may be written

$$\sum_{i=1}^{5}(a_i p_i + b_i q_i) = 0, \quad \sum_{i=1}^{5}(A_i p_i + B_i q_i) = 0, \tag{1}$$

the coefficients a_i, b_i, etc., being functions of the ten variables $x_1, y_1; \ldots; x_5, y_5$.

Two different numbers, i, j, say 1, 2, can now be found among 1, 2, 3, 4, 5, such that the equations (1) can be solved for two of the derivatives p_1, q_1, p_2, q_2. Suppose the derivatives thus selected are p_1, q_1. The equations (1) can then be written in the forms

$$\left.\begin{aligned} p_1 &= \alpha p_2 + \beta q_2 + \sum_{i=3}^{5}(\alpha_i p_i + \beta_i q_i), \\ q_1 &= \gamma p_2 + \delta q_2 + \sum_{i=3}^{5}(\gamma_i p_i + \delta_i q_i). \end{aligned}\right\} \tag{2}$$

As these differential equations are satisfied by all the two-point functions of the five points considered, the equations obtained from them by leaving out the terms $\alpha_5 p_5 + \beta_5 q_5$ and $\gamma_5 p_5 + \delta_5 q_5$ must be satisfied by all the two-point functions of the four points x_1, y_1; x_2, y_2; x_3, y_3; x_4, y_4, whatever may be the values of x_5 and y_5.

These last two variables may, or may not, be present in the coefficients of the equations (2) thus abridged. If they are present, we derive more than two independent differential equations that must be satisfied by the functions (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4). These six functions, containing eight variables, can therefore not all be independent. But then it is easily proved that the ten two-point functions of the five points $x_1, y_1; \ldots; x_5, y_5$, satisfy at least three relations, contrary to hypothesis.

Leaving out of (2) in the same manner the terms $\alpha_i p_i + \beta_i q_i$ and $\gamma_i p_i + \delta_i q_i$, $i = 3, 4$, we find that the coefficients α_i, etc., contain the variables $x_1, y_1, x_2, y_2, x_i, y_i$, only.

By definition, any one of the two-point functions is unaltered by interchanging the two points. The ten two-point functions determining the equations

(2) are therefore at most interchanged by interchanging any two of the five points considered. Subjecting the equations (2) to such a change would accordingly give us two new equations equivalent to the first two. It is thus immediately evident that α_i, β_i, etc., are the same functions of x_1, y_1, x_2, y_2, x_i, y_i, as α_j, β_j, etc., are of x_1, y_1, x_2, y_2, x_j, y_j, and that we could solve the equations (2) for two of the derivatives p_4, q_4, p_5, q_5.

Now, the two-point function $f(x_4, y_4;\, x_5, y_5)$, satisfying (2), must satisfy the following equations:

$$\left.\begin{aligned} \alpha_4 p_4 + \beta_4 q_4 + \alpha_5 p_5 + \beta_5 q_5 = 0, \\ \gamma_4 p_4 + \delta_4 q_4 + \gamma_5 p_5 + \delta_5 q_5 = 0, \end{aligned}\right\} \tag{3}$$

which are independent for general values of x_1, y_1, x_2 and y_2. Substituting some constant values for these variables that will not destroy the independence of the equations (3), we find that the function (4, 5) must satisfy two independent differential equations of the form

$$\left.\begin{aligned} c_4 p_4 + d_4 q_4 + c_5 p_5 + d_5 q_5 = 0, \\ C_4 p_4 + D_4 q_4 + C_5 p_5 + D_5 q_5 = 0, \end{aligned}\right\} \tag{4}$$

where the coefficients with the subscript 4 are functions of x_4 and y_4 *only*, and those with the subscript 5 the same functions with x_5 and y_5 in place of x_4 and y_4. We may remark that $c_4 D_4 - d_4 C_4 \neq 0$, otherwise it would follow from (4) that the ten two-point functions, (1, 2), ..., (4, 5) would satisfy the equations $c_i p_i + d_i q_i = 0$, $i = 1, 2, \ldots, 5$, in which case at most five of them could be independent, contrary to hypothesis.

Now, by the theory of partial differential equations, any solution common to the equations (4) must satisfy the equation obtained by forming the "commutator" of these two and putting it equal to zero,

$$\begin{aligned} &\left(c_4 \frac{\partial}{\partial x_4} + d_4 \frac{\partial}{\partial y_4} + c_5 \frac{\partial}{\partial x_5} + d_5 \frac{\partial}{\partial y_5}\right)\left(C_4 p_4 + D_4 q_4 + C_5 p_5 + D_5 q_5\right) \\ &\quad - \left(C_4 \frac{\partial}{\partial x_4} + D_4 \frac{\partial}{\partial y_4} + C_5 \frac{\partial}{\partial x_5} + D_5 \frac{\partial}{\partial y_5}\right)\left(c_4 p_4 + d_4 q_4 + c_5 p_5 + d_5 q_5\right) = 0. \end{aligned} \tag{5}$$

The left-hand member is readily seen to be the commutator of $c_4 p_4 + d_4 q_4$ and $C_4 p_4 + D_4 q_4$ plus the commutator of $c_5 p_5 + d_5 q_5$ and $C_5 p_5 + D_5 q_5$. The new equation is, therefore, of the same form as the equations (4).

New differential equations may now be formed from this one and each of the equations (4), and so on. As all the differential equations so obtained have a solution (4, 5) in common, at most three of them can be independent. If three of them are independent, the ten two-point functions of the five points $x_1, y_1; \ldots; x_5, y_5$ are connected by at least three relations. For, let $\alpha'_4 p_4 + \beta'_4 q_4 + \alpha'_5 p_5 + \beta'_5 q_5 = 0$ be any one of the equations satisfied by (4, 5), and it is clear that the differential equation

$$\alpha'_1 p_1 + \beta'_1 q_1 + \alpha'_2 p_2 + \beta'_2 q_2 + \alpha'_3 p_3 + \beta'_3 q_3 + \alpha'_4 p_4 + \beta'_4 q_4 + \alpha'_5 p_5 + \beta'_5 q_5 = 0$$

is satisfied by all the ten two-point functions considered. We should have at least three such equations, independent of each other, and, therefore, at most $10 - 3 = 7$ independent common solutions.

Consider next the case where only two of the differential equations of the form

$$\alpha'_4 p_4 + \beta'_4 q_4 + \alpha'_5 p_5 + \beta'_5 q_5 = 0$$

are independent. The differential equation (5) should here be derivable from the equations (4). Applying this condition, we find the following identity in p_4 and q_4:

$$\left(c_4 \frac{\partial}{\partial x_4} + d_4 \frac{\partial}{\partial y_4}\right)\left(C_4 p_4 + D_4 q_4\right) - \left(C_4 \frac{\partial}{\partial x_4} + D_4 \frac{\partial}{\partial y_4}\right)\left(c_4 p_4 + d_4 q_4\right)$$
$$\equiv \phi_5 (c_4 p_4 + d_4 q_4) + \psi_5 (C_4 p_4 + D_4 q_4),$$

ϕ_5 and ψ_5 being functions of x_5 and y_5 only. Since none other of the coefficients involve x_5 and y_5, we can take some general constant values of these and obtain the result that the commutator of $c_4 p_4 + d_4 q_4$ and $C_4 p_4 + D_4 q_4$ is of the form

$$m (c_4 p_4 + d_4 q_4) + n (C_4 p_4 + D_4 q_4),$$

where m and n are constants. But this is evidently the necessary and sufficient condition that the two expressions

$$c_4 p_4 + d_4 q_4, \quad C_4 p_4 + D_4 q_4, \tag{6}$$

regarded as *infinitesimal transformations* in Lie's group theory, generate a two-

parametric group of point transformations in two variables. By a transformation of the variables x_4, y_4 of the form

$$x_4 = \phi(x, y), \quad y_4 = \psi(x, y),$$

the two expressions (6) can be changed into one of the following two types:*

$$p, q; \quad p, \quad xp + yq. \tag{7}$$

We have thus the theorem:

1. *If the two-point function* (i, j), $i \neq j$, *is of such a nature that by selecting any five points* $x_1, y_1; \ldots ; x_5, y_5$, *the ten functions* (i, j), $i, j = 1, 2, \ldots, 5$; $i \neq j$, *are connected by just two relations independent of the coordinates of these five points, then must the function* (i, j) *be a two-point invariant of a continuous group in two variables similar to one or other of the groups* (7).

Conversely, if a given two-point function is a two-point invariant of such a group, then must the ten two-point functions of any five points be connected by at least two relations, as they satisfy two independent linear partial differential equations.

II.—*The Ten Two-Point Functions of Five Points are Connected by just Three Relations.*

§2.

In this case we have three differential equations corresponding to the equations (1). Two cases may now present themselves as follows:

(A). The three equations can be solved for three of the derivatives p_1, q_1, p_2, q_2.

(B). From the three equations can be eliminated the derivatives p_1, q_1, p_2, q_2.

In (A) we can proceed as above in the case 1. We get three independent equations corresponding to the equations (4), from which we build commutators and get equations of the form (5). The condition that the latter should be derivable from the former gives two different results, found by equating to zero

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* See Lie's "Differentialgleichungen mit bekannten infinitesimalen Transformationen," page 425. The types p, xp; q, xq are omitted here, as in our groups the determinant $c_4 D_4 - C_4 d_4 \neq 0$.

the determinants of the coefficients of all the equations obtained. These results are:

1°. The two-point function (4, 5) satisfies a differential equation of the form

$$s_4 p_4 + t_4 q_4 = 0, \tag{8}$$

where s_4 and t_4 are functions of x_4 and y_4 only.

2°. The two-point functions are two-point invariants of a three-parametric continuous group in two variables.

In case 1°, the ten two-point functions of five points must evidently satisfy the five independent differential equations

$$s_i p_i + t_i q_i = 0, \qquad i = 1, 2, \ldots, 5.$$

They must, therefore, satisfy at least five relations, contrary to hypothesis.

In the case 2°, no two of the independent infinitesimal transformations of the group considered can differ only by a factor. For, let $ap + bq$, $\phi(ap + bq)$ be two such transformations. The functions (4, 5) must then satisfy the equations

$$a_4 p_4 + b_4 q_4 + a_5 p_5 + b_5 q_5 = 0, \quad \phi_4 (a_4 p_4 + b_4 q_4) + \phi_5 (a_5 p_5 + b_5 q_5) = 0,$$

from which we get

$$a_4 p_4 + b_4 q_4 = 0, \quad a_5 p_5 + b_5 q_5 = 0,$$

equations of the form considered in 1°.

The three-parametric groups in two variables having no such pairs of infinitesimal transformations are of the following types:*

$$\left.\begin{array}{l} p, \quad q, \quad xp + cyq, \quad c \neq 0; \\ p + x^2 p + xyq, \quad q + xyp + y^2 q, \quad yp - xq; \\ xq, \quad xp - yq, \quad yp; \\ p, \quad q, \quad xp + (x + y) q. \end{array}\right\} \tag{9}$$

It is readily found that the two-point invariants of any one of these groups can be written in such a form as to satisfy all the requirements of the two-point functions under consideration.

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* "Theorie der Transformationsgruppen," Lie-Engel, Vol. III, p. 57. The types here given are taken from the list on p. 436.

Let us now consider case (B). One of the three differential equations satisfied by the ten functions (1, 2), ..., (4, 5), can be written

$$s_3 p_3 + t_3 q_3 + s_4 p_4 + t_4 q_4 + s_5 p_5 + t_5 q_5 = 0.$$

Suppose $s_4 \neq 0$. The two functions (1, 4) and (2, 4) must evidently satisfy the differential equation

$$p_4 + \frac{t_4}{s_4} q_4 = 0. \tag{10}$$

It follows that $\frac{t_4}{s_4}$ is a function of x_4, y_4 only, unless the two-point functions are constants merely. If, however, $\frac{t_4}{s_4}$ is a function of x_4, y_4 only, the equation (10) is of the same form as (8) in 1°, an impossible case.

To resume, we have the theorem:

2. *The necessary and sufficient condition that the ten two-point functions of any five points satisfy just three relations is that they are two-point invariants of a three-parametric continuous group in two variables similar to one or other of the types* (9).

By examining the invariants of the groups considered we find that *the six two-point functions of the kind here defined in any four points satisfy just one relation.*

III.—*The Ten Two-Point Functions of any Five Points Satisfy at least Four Relations.*

§3.

In the first place we find, by considering the possible sets of four relations connecting the ten two-point functions (1, 2), ..., (4, 5), that one relation at least must connect the six functions (1, 2), ..., (3, 4) of any four points. Then we find that one relation at least must exist between the functions (1, 2), (1, 3), (2, 3); (1, 4), (2, 4); (1, 5), (2, 5). If (2, 5) is actually present in this relation, we may replace x_1, y_1, x_3, y_3, x_4, y_4 by arbitrary constants, and change the variables so that (1, 5) becomes the new x_5. We thus obtain (2, 5) as a function of x_5, x_2, y_2, which must evidently reduce to a function of x_5, x_2 only; say $(2, 5) \equiv f(x_2, x_5)$. The ten two-point functions of any five points will obviously satisfy at least five relations.

In the same manner all possible cases of having one relation connect the functions (1, 2), (1, 3), (2, 3); (1, 4), (2, 4); (1, 5), (2, 5) may be considered. The same result will be obtained.

If more than five relations connect the ten two-point functions of five points, we find that one relation at least connects the three two-point functions of three points. Suppose there is one such relation. Let the three functions be

$$f(x_1, x_2), \quad f(x_2, x_3), \quad f(x_3, x_1),$$

and, by our usual process of building a linear partial differential equation, etc., we find that this can, by a suitable choice of variables, be put into the form

$$p_1 + p_2 + p_3 = 0.$$

The two-point function (1, 2) is, accordingly, of the type $(x_1 - x_2)^2$. Additional relations existing among the two-point functions considered will cause these to be constants merely, as is readily proved.

Accordingly, we have the theorem:

3. *If at least four relations connect the ten two-point functions of any five points, these two-point functions are, by a proper choice of variables, reducible to the form*

$$f(x_i, x_j).$$

Conversely, if the two-point functions have this form, at least five relations connect the ten two-point functions of five points.

More than five relations connecting these functions could exist only if the two-point functions, which are not supposed to be constants merely, are reducible to the type

$$(x_i - x_j)^2.$$

By consulting the theorems 1–3, observing that each of the groups (9) contains two-parametric subgroups similar to one or other of the groups (7), we can make the following statement:

4. *The necessary and sufficient condition that the ten two-point functions of any five points are connected by two or more relations, is that the two-point function* (i, j), $i \neq j$, *is reducible to the form* $f(x_i, x_j)$ *by a change of variables, or that the two-point functions are two-point invariants of a continuous group in two variables similar to one or other of the groups*

$$p, q; \quad p, xp + yq.$$

Thus, by a proper choice of the variables x, y, the two-point functions possessing the property just stated are represented by the three different types

$$f(x_i, x_j); \quad f(x_i - x_j, y_i - y_j); \quad f\left(\frac{x_i - x_j}{y_i}, \frac{x_i - x_j}{y_j}\right).$$

SOME GEOMETRICAL APPLICATIONS.

I.—*Concerning Angles.*

§4.

Certain functions of the angles

$$\cos^{-1} \frac{a_1 a_2 + b_1 b_2 + 1}{\sqrt{1 + a_1^2 + b_1^2}\sqrt{1 + a_2^2 + b_2^2}} \quad \text{and} \quad \tan^{-1} \frac{m_1 - m_2}{1 + m_1 m_2}$$

between the two planes $a_1 x + b_1 y + z = c_1$, $a_2 x + b_2 y + z = c_2$ and the two straight lines $y + m_1 x = b_1$, $y + m_2 x = b_2$ respectively, as

$$\frac{(a_1 a_2 + b_1 b_2 + 1)^2}{(1 + a_1^2 + b_1^2)(1 + a_2^2 + b_2^2)} \quad \text{and} \quad \frac{(m_1 - m_2)^2}{(1 + m_1 m_2)^2},$$

are types of two-point functions.

The first, written in the variables x, y in place of a, b, is a two-point invariant of the second of the groups (9). The six angles made by four planes in space are connected by one relation.

The second may be written $[\tan(\tan^{-1} m_1 - \tan^{-1} m_2)]^2$, and is one of the types of two-point functions considered in §3. The three angles made by three straight lines in a plane are connected by one relation.

II.—*Concerning Distances.*

§5.

The two-point functions which are such that just one relation exists between the six two-point functions of four points must, as has been demonstrated, be two-point invariants of continuous groups similar to the group-types (9). Replacing the first of these types by the similar group

$$p, \quad q, \quad yp - xq + k(xp + yq),$$

the two-point invariants of the two points x_1, y_1; x_2, y_2 are respectively for the four groups

$$\left.\begin{array}{ll} \text{A.} & \{(x_2-x_1)^2+(y_2-y_1)^2\}\, e^{2k\tan^{-1}\frac{y_2-y_1}{x_2-x_1}}; \\ \text{B.} & \dfrac{(x_2-x_1)^2+(y_2-y_1)^2+(x_1y_2-x_2y_1)^2}{(1+x_1x_2+y_1y_2)^2}; \\ \text{C.} & (x_1y_2-x_2y_1)^2; \\ \text{D.} & (x_2-x_1)^2\, e^{-2\frac{y_2-y_1}{x_2-x_1}}. \end{array}\right\} \tag{11}$$

Denoting, as above, by (a, b), the two-point function of the two points x_a, y_a; x_b, y_b, the relations connecting (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), for the different types given, are found by elimination of the coordinates x_1, y_1, etc. They are, respectively,

A. The result of eliminating s and t between the equations

$$[23](s-1)^c-[13]s^c+[12]=0,$$
$$[24](t-1)^c-[14]t^c+[12]=0,$$
$$[34](t-s)^c-[14]t^c+[13]s^c=0,$$

where $c=\dfrac{k-i}{k+i}$, $i=\sqrt{-1}$; and $[ab]=(a, b)^{\frac{1-c}{2}}$;

B. $1+2[23][24][34]=[23]^2+[24]^2+[34]^2$,

where $$[ab]=\frac{\overline{(a, b)}-\overline{(1, a)}\,\overline{(1, b)}}{\{(\overline{(1, a)}^2-1)(\overline{(1, b)}^2-1)\}^{\frac{1}{2}}},$$

and $$\overline{(a, b)}=\frac{1}{\sqrt{(a, b)+1}}=\frac{1+x_1x_2+y_1y_2}{\sqrt{1+x_1^2+y_1^2}\,\sqrt{1+x_2^2+y_2^2}}; \tag{12}$$

C. $[12][34]+[13][42]+[14][23]=0$,

where $[ab]=(a, b)^{\frac{1}{2}}$;

D. The result of eliminating s and t between the equations

$$(s-1)\log(s-1)-s\log s=(s-1)[23]-s[13]+[12],$$
$$(t-1)\log(t-1)-t\log t=(t-1)[24]-t[14]+[12],$$
$$(t-s)\log(t-s)-t\log t+s\log s$$
$$=(t-s)[34]-t[14]+s[13],$$

where $[ab]=\frac{1}{2}\log(a, b)$.

The distance between two points in the plane is a two-point function of the kind here considered. Thus we see that the property of the analytic expression for the distance between two points of remaining unaltered by exchanging the coordinates of the two points, added to the property that the six distances connecting any four points are connected by just one relation, limits this expression to a function of one of the four types (11).

Examining more closely the two-point invariants (11) and the relations (12) we get the following result:*

The distance between two points, or a function of this distance, in the euclidean or non-euclidean plane, is determined by the following conditions:

The distance is a two-point function, which is real if real coordinates are used. Through any general point in the plane no real curve passes, the distance from every point of which to the given point is indeterminate, a constant, or is infinitely great. The six distances connecting any four points in the plane are connected by just one relation, which is an algebraic relation between certain functions of these distances.

To apply these conditions we have first to find what values of the arbitrary constant k and what transformations of the coordinates involved will render certain functions of the invariants (11) real quantities for real coordinates. To answer this question it is plainly sufficient to find all the *real group types*† similar to the types (9).

These types are given in Lie's "Theorie der Transformationsgruppen," Vol. III, p. 436. To the four given in (9), assumed to contain real elements, must be added the following two:

$$p, \quad q, \quad yp - xq + k(xp + yq);$$
$$p - x(xp + yq), \quad q - y(xp + yq), \quad yp - xq;$$

also containing real elements. These groups are similar to the first two of the groups (9), and their two-point invariants (1, 2) are respectively (A) of (11) and

$$\text{E.} \quad \frac{(x_1 y_2 - x_2 y_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2}{(1 - x_1 x_2 - y_1 y_2)^2}.$$

The two-point invariant (1, 2) of the first of the groups (9) is

$$\text{F.} \quad (y_2 - y_1)^2 (x_2 - x_1)^{-2c}.$$

* Compare "On the determination of the distance between two points in space of n dimensions," by the author, *Transactions of the Am. Math. Soc.*, Oct., 1903, p. 467.

† Cf. "Theorie der Transformationsgruppen," Vol. III, Chapter 19, for real groups.

Applying the second condition given above to the six invariants obtained, we find that the invariant (F) just given and the two last, (C) and (D) of (11), must be excluded having respectively the curves

$$y = \text{constant}, \quad \frac{y}{x} = \text{constant}, \quad x = \text{constant},$$

of the kind excluded.

The types (A), (B) and (E) must now be considered. The two last fulfill all the conditions given, and are, in fact, functions ($\tan^2 d$ and $\tan^2 id$) of the distance d between two points in the elliptic and hyperbolic plane respectively, proper coordinates being chosen. In the type (A), k must be real by the first condition, and by referring to the relations connecting the six invariants (1, 2), etc., given under (A) in (12), we find that c must be real in order that the relation connecting these six invariants may be algebraic. Since $c = \frac{k-i}{k+i}$, $i = \sqrt{-1}$, from which $k = i\,\frac{1+c}{1-c}$, the only permissible value for k is zero. (If $k = \infty$, the type (A) of (11) will reduce to a particular case of type (F).)

§6.

III.—*Problem: What are the surfaces for which, any five points be taken and joined by chords, the lengths of the ten chords so obtained are connected by two or more relations independent of the location of the points on the surface?*

Let such a surface be given by $z = f(x, y)$, and let us write p, q, r, s, t for $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$, as is customary. To avoid confusion, we shall henceforth write $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for p, q in the infinitesimal transformations concerned, reserving the letters p, q for $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

The square of the distance connecting the points x_1, y_1, z_1 and x_2, y_2, z_2 is

$$(1, 2) \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

where $$z_1 = f(x_1, y_1), \quad z_2 = f(x_2, y_2).$$

According to the theorem (4), §3, the two-point function (1, 2) must be a two-point invariant of a continuous group similar to one or other of the groups

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}; \quad \frac{\partial f}{\partial x}, \ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y};$$

or it must, by a change of variables, be reducible to the form $\phi(x_1, x_2)$.

If the latter is the case, we must have the identity

$$\phi(\alpha_1, \alpha_2) \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

where α_1 is a function of x_1 and y_1, only, α_2 the same function of x_2 and y_2.

Differentiating this equation in turn by x_1 and y_1, and eliminating $\frac{\partial \phi}{\partial \alpha_1}$, we obtain the equation

$$(z_2 q_1 + y_2 - z_1 q_1 - y_1)\frac{\partial \alpha_1}{\partial x_1} + (-z_2 p_1 - x_2 + z_1 p_1 + x_1)\frac{\partial \alpha_1}{\partial y_1} = 0,$$

where $p_1 = \frac{\partial z_1}{\partial x_1}$, etc.

If we now give to x_1 and y_1 general constant values, we get the equation $z_2 a + y_2 b + x_2 c + d = 0$, a, b, etc., being constants, at least one of which is different from zero, as $\frac{\partial \alpha_1}{\partial x_1}$ and $\frac{\partial \alpha_1}{\partial y_1}$ cannot both be zero identically. The surface is, in such a case, a plane, for which the conditions of the problem are, a priori, satisfied.

Thus it remains for us to find the surfaces $z = f(x, y)$ for which the function (1, 2) is a two-point invariant of a group similar to one or other of the groups

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}; \quad \frac{\partial f}{\partial x}, \ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

Let the infinitesimal transformations of such a group be

$$\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y}, \quad \gamma\frac{\partial f}{\partial x} + \delta\frac{\partial f}{\partial y}.$$

In order that the function (1, 2) be a two-point invariant of this group, it should satisfy the differential equations

$$\alpha_1\frac{\partial f}{\partial x_1} + \beta\frac{\partial f}{\partial y_1} + \alpha_2\frac{\partial f}{\partial x_2} + \beta_2\frac{\partial f}{\partial y_2} = 0, \quad \gamma_1\frac{\partial f}{\partial x_1} + \text{etc.} = 0,$$

α_1 being written in the variables x_1, y_1; α_2 in x_2, y_2, etc.

That the first of these equations may be satisfied by (1, 2), we must have identically

$$(x_1-x_2)(\alpha_1-\alpha_2)+(y_1-y_2)(\beta_1-\beta_2)$$
$$+(z_1-z_2)(\alpha_1 p_1+\beta_1 q_1-\alpha_2 p_2-\beta_2 q_2)\equiv 0\,. \qquad (13)$$

By substituting in turn different sets of constant values for x_2 and y_2 in this equation, we can solve for $\alpha_1 p_1+\beta_1 q_1$, α_1 and β_1 linearly in terms of x_1, y_1 and z_1, unless the surface sought is a plane.

Disregarding the latter case (an obvious solution of the problem), we find the identity (13) satisfied by

$$\alpha=-az+ky+g\,,\quad \beta=-bz-kx+h\,,\quad \alpha p+\beta q=ax+by+c\,,$$

a, b, c, k, g, h being constants.

Similarly,

$$\gamma=-Az+Ky+G\,,\quad \delta=-Bz-Kx+H\,,\quad \gamma p+\delta q=Ax+By+C\,.$$

The equations

$$\alpha p+\beta q=ax+by+c\,,\quad \gamma p+\delta q=Ax+By+C$$

(being made consistent by modifying the constants a, b, etc.), will now determine z. Restricting ourselves to real solutions, the following surfaces only satisfy the conditions of the problem under consideration:

1°. Any series of a finite number of parallel planes.
2°. Any series of a finite number of concentric spheres.
3°. Any series of a finite number of co-axial right circular cylinders.

The mutual distances connecting *four* points on either of the first two surfaces are found without much difficulty to be connected by one relation, whereas, in the remaining surface, the mutual distances connecting five points are bound by just two relations. Thus, *the surface consisting of any series of co-axial right circular cylinders is the only real surface for which, any five points being taken, the mutual distances (along chords) connecting these five points are connected by just two relations.*

§7.

IV.—*Problem: What are the surfaces for which, any five points being taken and joined by geodesics, the ten geodesic distances so obtained are connected by two or more relations independent of the positions of the points?*

Before going into the details of this problem, we may remark that the six mutual geodesic distances of four points on a surface of constant curvature are connected by one relation.

Any two such surfaces with the same curvature are, as is well known, applicable one upon the other. As representative surfaces, the sphere $x^2+y^2+z^2=a_2$ with curvature $\frac{1}{a^2}$ and the pseudosphere*

$$x^2+y^2=a^2\sin^2\phi, \quad z=a\left(\log\tan\frac{\phi}{2}+\cos\phi\right)$$

with curvature $-\frac{1}{a^2}$ may therefore be selected. The geodesic distance d between the two points x_1, y_1, z_1; x_2, y_2, z_2 are respectively given by

$$\sin^2\frac{d}{2a}=\frac{(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2}{-4\beta_1\beta_2}, \quad \sin^2\frac{d}{2ia}=\frac{(\alpha_2-x_1)^2+(\beta_2-\beta_1)^2}{-4\beta_1\beta_2},$$

where α and β are certain functions of x, y and z.†

Now, the distances connecting four points on a sphere are, as we know, connected by one relation. The distances on the pseudosphere in question being of the same type as the distances on the sphere, differing from these only by the multiplier i after a proper choice of coordinates, it is evident that one relation must connect the six mutual distances of four points on the pseudosphere.

Now, we shall prove that the surfaces required in the present problem, if real, must have constant curvature.

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* Darboux, "Theorie des Surfaces," T. III, p. 394.

† Darboux, "Theorie des Surfaces," T. III, p. 401. The variables x, y used by M. Darboux in the formulæ referred to are here replaced by α and β respectively.

Under the conditions of the problem, the distance (1, 2) must be a two-point invariant of a continuous group similar to one of the groups

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}; \quad \frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \tag{14}$$

or it must be reducible to the form $\phi(x_1, x_2)$ by a change of variables (theorem 4, §3).

For a real surface, the distance (1, 2) could not be of the form $\phi(x_1, x_2)$. This would, namely, require the linear element ds of the surface to be rational in dx, as is easily seen. The linear element of a real surface $z = f(x, y)$,

$$ds = \sqrt{\{(1+p^2)\,dx^2 + 2pq\,dx\,dy + (1+q^2)\,dy^2\}}$$

could, however, not become rational in dx and dy by any change of variables.

Thus the distance (1, 2) must be a two-point invariant of a group similar to one or other of the groups (14). Taking the two points indefinitely near each other, and writing x, y, $x + dx$, $y + dy$ for x_1, y_1, x_2, y_2 respectively, we have

$$(1, 2)^2 = ds^2 = (1+p^2)\,dx^2 + 2pq\,dx\,dy + (1+q^2)\,dy^2.$$

Introducing in this expression the variables x, y used in the groups (14) we obtain

$$ds^2 = E dx^2 + 2F\,dx\,dy + G dy^2,$$

E, F, G being functions of x and y. In order that this expression may be unaltered by the point transformation determined by one or other of the groups (14), ds^2 must satisfy one or other of the following systems of simultaneous partial differential equations in x, y, dx, dy:

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0,$$

$$\frac{\partial f}{\partial x} = 0, \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + dx\frac{\partial f}{\partial\, dx} + dy\frac{\partial f}{\partial\, dy} = 0,$$

The solutions are respectively

$$a dx^2 + 2b dx\,dy + c dy^2, \quad \frac{1}{y^2}(a dx^2 + 2b dx\,dy + c dy^2),$$

a, b and c being constants.

Linear transformations will reduce these expressions to the forms

$$dx^2 + dy^2, \quad -\frac{dx^2 + dy^2}{ky^2},$$

which are representative forms for the linear element of surfaces of constant curvature, zero and k respectively. Hence,

If two relations exist between the ten mutual geodesic distances of any five points on a real surface, independent of the coordinates of the five points, this must be a surface of constant curvature. On such a surface one relation exists between the six mutual geodesic distances of any four points.

STANFORD UNIVERSITY, CAL., *April*, 1902.

Surfaces whose Lines of Curvature in one System are Represented on the Sphere by Great Circles.

By L. P. Eisenhart.

Guichard* has shown that the determination of all congruences whose developables have a given spherical representation, reduces to the solution of an equation of Laplace, after the direction cosines have been found. We apply this method to the case where one family of lines on the sphere is composed of great circles and the other consists of their orthogonal trajectories. It is evident that all of these congruences are normal, and that the lines on the sphere are the images of the lines of curvature of the parallel surfaces which cut the lines orthogonally. This furnishes a means for the determination and study of surfaces whose lines of curvature in one system are represented on the sphere by great circles.

In the first place, we find that when such a configuration is given upon the sphere, the determination of the direction cosines of the lines of the congruence is the same problem as the finding of a skew curve from its intrinsic equations. The equation of Laplace for this special case can be solved by two quadratures, and as two arbitrary functions are introduced, there is a double infinity of families of parallel surfaces whose lines of curvature have the given representation.

It is found that these surfaces are characterized by the property that one family of the lines of curvature are geodesics, and along the lines of curvature of the second system one of the principal radii is constant. From these properties it follows that one of the sheets of the evolute is a developable surface, and

* "Surfaces rapportées à leurs lignes asymptotiques et congruences rapportées à leurs developpables" (Annales Scientifiques de l'École Normale Supérieure, t. VI, 3e serie).

conversely. Hence, the surfaces under consideration are the *surfaces of Monge*,* and consequently can be generated by a plane curve whose plane rolls without sliding upon a developable surface. It is shown that one of the arbitrary functions, which appear in the expression for the semi-focal distance, depends entirely upon the generating curve, and that the other function in connection with the spherical representation determines the character of the generating developable. In particular, the *moulure* surfaces are considered from this point of view.

The second sheet of the evolute is a surface of Monge with the same surface generator, and its generating curve is the evolute of the curve for the surface. These curves are represented on the sphere by the same great circle and corresponding points are at the distance of a quadrant.

Surfaces of revolution form a subclass of moulure surfaces, corresponding to the case where the surface generator is a straight line. It is evident that there are only particular families of great circles which can be the images of the meridians. For these systems the direction cosines can be found by quadratures, and hence the complete determination of all surfaces of revolution, whose meridians have a given representation, reduces to quadratures.

Finally, we show that surfaces of revolution are the only surfaces of Monge which are Weingarten surfaces, and that they are the only isothermic surfaces of Monge.

1. Consider a sphere of radius unity and with center at the origin of coordinates, and let X, Y, Z denote the cartesian coordinates of a point on the sphere, or, what is the same thing, the direction cosines of the radius. Let the sphere be referred to a system of lines $v=$ const., $u=$ const., and write

$$\mathcal{E}=\Sigma\left(\frac{\partial X}{\partial u}\right)^2,\quad \mathcal{F}=\Sigma\frac{\partial X}{\partial u}\,\frac{\partial X}{\partial v},\quad \mathcal{G}=\Sigma\left(\frac{\partial X}{\partial v}\right)^2. \tag{1}$$

In particular, we consider the case where the curves $u=$ const. are great circles and $v=$ const. are their orthogonal trajectories; then $\mathcal{F}$ is zero and $\mathcal{G}$ is a function of v alone By a proper choice of parameters we can have

$$\mathcal{F}=0,\quad \mathcal{G}=1. \tag{2}$$

* Application de l'Analyse à la Géométrie, 5me edition, p. 322.

Since $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ cannot be chosen arbitrarily, but must satisfy the Gauss equation*

$$\frac{\partial}{\partial u}\left[\frac{\mathfrak{F}}{\mathfrak{E}\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial \mathfrak{E}}{\partial v}-\frac{1}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial \mathfrak{G}}{\partial u}\right]$$
$$+\frac{\partial}{\partial v}\left[\frac{2}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial \mathfrak{F}}{\partial u}-\frac{1}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial \mathfrak{E}}{\partial v}-\frac{\mathfrak{F}}{\mathfrak{E}\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial \mathfrak{E}}{\partial u}\right]$$
$$=2\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}, \tag{3}$$

which, for the present case, reduces to

$$\frac{\partial^2\sqrt{\mathfrak{E}}}{\partial v^2}+\sqrt{\mathfrak{E}}=0,$$

we find that $\sqrt{\mathfrak{E}}$ is of the form

$$\sqrt{\mathfrak{E}}=U_1\cos v+U_2\sin v, \tag{4}$$

where U_1, U_2 are arbitrary functions of u alone.

Denote by λ_1, μ_1, ν_1; λ_2, μ_2, ν_2, the direction cosines of the tangents to the curves $v=\text{const.}$, $u=\text{const.}$ respectively on the sphere. They satisfy the following relations:†

$$d\lambda_2=\frac{\partial\sqrt{\mathfrak{E}}}{\partial v}\lambda_1\,du-X\,dv,\quad dX=\sqrt{\mathfrak{E}}\lambda_1\,du+\lambda_2\,dv, \tag{5}$$

and similarly for Y, μ_1, μ_2 and Z, ν_1, ν_2. From the last two we have

$$\frac{\partial\lambda_2}{\partial v}+X=0,\quad \frac{\partial X}{\partial v}=\lambda_2, \tag{6}$$

whence

$$\frac{\partial^2 X}{\partial v^2}+X=0,$$

so that

$$X=U_{11}\cos v+U_{12}\sin v, \tag{7}$$

and in like manner

$$\left.\begin{aligned}Y&=U_{21}\cos v+U_{22}\sin v,\\ Z&=U_{31}\cos v+U_{32}\sin v,\end{aligned}\right\} \tag{7'}$$

where $U_{11}, \ldots, U_{32}$ are functions of u alone. Since $\Sigma X^2=1$, these functions

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* Bianchi, Lezioni, p. 67.

† Bianchi, Lezioni, p. 94.

must satisfy the conditions

$$U_{11}^2 + U_{21}^2 + U_{31}^2 = 1, \quad U_{12}^2 + U_{22}^2 + U_{32}^2 = 1,$$
$$U_{11}U_{12} + U_{21}U_{22} + U_{31}U_{32} = 0. \tag{8}$$

Again from (5) we have

$$\frac{\partial \lambda_2}{\partial u} = \frac{\partial \sqrt{\mathfrak{G}}}{\partial v}\lambda_1, \quad \frac{\partial X}{\partial u} = \sqrt{\mathfrak{G}}\,\lambda_1;$$

eliminating λ_1 and replacing $\sqrt{\mathfrak{G}}$, X, λ_2 by their expressions, we get

$$U_1U'_{12} - U_2U'_{11} = 0, \quad U_1U'_{22} - U_2U'_{21} = 0, \quad U_1U'_{32} - U_2U'_{31} = 0, \tag{9}$$

where the accents denote differentiation with respect to u.

In the first of (1), substitute for $\mathfrak{G}$, X, Y, Z, their expressions from (4), (7); then

$$(U_1 \cos v + U_2 \sin v)^2 = \Sigma\,(U'_{11} \cos v + U'_{12} \sin v)^2,$$

and, consequently,

$$U_1^2 = \Sigma\, U'^2_{11}, \quad U_2^2 = \Sigma\, U'^2_{12}, \quad U_1U_2 = \Sigma\, U'_{11}U'_{12}.$$

Combining these results with (9), we find

$$\left.\begin{aligned}\frac{U'_{11}}{U_{21}U_{32} - U_{31}U_{22}} = \frac{U'_{21}}{U_{31}U_{12} - U_{11}U_{32}} = \frac{U'_{31}}{U_{11}U_{22} - U_{21}U_{12}} = U_1,\\ \frac{U'_{12}}{U_{21}U_{32} - U_{31}U_{22}} = \frac{U'_{22}}{U_{31}U_{12} - U_{11}U_{32}} = \frac{U'_{32}}{U_{11}U_{22} - U_{21}U_{12}} = U_2.\end{aligned}\right\} \tag{10}$$

These conditions, (8), (10), which the six functions $U_{11}, \ldots, U_{32}$ must satisfy in order that the expressions (7) shall be the coordinates of a point on the sphere of radius unity for which the curves $u =$ const. are great circles, are the very conditions which these functions would necessarily satisfy if U_{11}, U_{21}, U_{31} were the direction cosines of the tangent and U_{12}, U_{22}, U_{32} of the binormal of a curve defined by the intrinsic equations

$$\rho = \frac{1}{U_1}, \quad \tau = \frac{1}{U_2},$$

where ρ and τ are the radii of curvature and torsion respectively. Hence, *the*

problem of finding the direction cosines of the lines of a congruence, whose developables are represented on the sphere by a given family of great circles and their orthogonal trajectories, is equivalent to the determination of a skew curve from its intrinsic equations.

When the coordinates of a point on a sphere are given by (7), where $U_{11}, \ldots, U_{32}$ satisfy the conditions (8), the curves $u=$ const. on the sphere are great circles and $v=$ const. are their orthogonal trajectories. Hence the inverse of the above problem reduces to the determination of functions satisfying the conditions (8).

2. For the special spherical system which we are considering, the equation of Laplace, found by Guichard* and which is satisfied by the semi-focal distance for any of the congruences with the given representation of their developables, reduces to the form

$$\frac{\partial^2 \rho}{\partial u \, \partial v} + \frac{\partial \log \sqrt{\mathfrak{G}}}{\partial v} \frac{\partial \rho}{\partial u} + \frac{\partial^2}{\partial u \, \partial v} \log \sqrt{\mathfrak{G}} \, . \, \rho = 0. \tag{11}$$

By one quadrature we get

$$\frac{\partial \rho}{\partial v} + \frac{\partial \log \sqrt{\mathfrak{G}}}{\partial v} \rho = V, \tag{11'}$$

where V is a function of v alone, and by a second quadrature

$$\rho = \frac{1}{\sqrt{\mathfrak{G}}} \left(\int \sqrt{\mathfrak{G}} \, V dv + U \right), \tag{12}$$

where U is a function of u alone. Since it can be shown that to every solution of the general equation found by Guichard there corresponds a congruence with the given representation; it follows that the functions U and V are perfectly arbitrary in the present case.

Guichard has shown that the cartesian coordinates, x_1, y_1, z_1, of a point on the middle surface of the congruences corresponding to a particular solution of equation (11), are given by†

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* Bianchi, Lezioni, p. 262.

† Bianchi, p. 262.

$$\left.\begin{aligned}\frac{\partial x_1}{\partial u} &= \frac{\partial \rho}{\partial u} X - \rho \frac{\partial X}{\partial u}, \\ \frac{\partial x_1}{\partial v} &= -\left(\frac{\partial \rho}{\partial v} + \frac{\partial \log \mathfrak{G}}{\partial v} \rho\right) X + \rho \frac{\partial X}{\partial v},\end{aligned}\right\} \tag{13}$$

and similar expressions in y_1 and z_1.

3. We consider now the congruence corresponding to a given function $\sqrt{\mathfrak{G}}$ and a particular form of ρ, and let S denote one of the surfaces orthogonal to this congruence. Denote by x, y, z the cartesian coordinates of a point on S and by r the distance of the point from the corresponding point on the mean surface; then

$$x = x_1 + rX, \quad y = y_1 + rY, \quad z = z_1 + rZ. \tag{14}$$

Differentiating these equations and multiplying by X, Y, Z respectively, we find

$$dr = -\Sigma X \, dx_1, \tag{15}$$

or

$$\frac{\partial r}{\partial u} = -\Sigma X \frac{\partial x_1}{\partial u}, \quad \frac{\partial r}{\partial v} = -\Sigma X \frac{\partial x_1}{\partial v}. \tag{15'}$$

When the expressions for $\frac{\partial x_1}{\partial u}, \ldots, \frac{\partial z_1}{\partial v}$, as given by (13), are substituted in these equations, they reduce to

$$\frac{\partial r}{\partial u} = -\frac{\partial \rho}{\partial u}, \quad \frac{\partial r}{\partial v} = \frac{\partial \rho}{\partial v} + \frac{\partial \log \mathfrak{G}}{\partial v} \rho. \tag{16}$$

From the first of these we get

$$r = -\rho + V_2,$$

where V_2 is a function of v alone. Substituting this expression in the second of (16), we find that V_2 satisfies the condition

$$V_2' = 2\left(\frac{\partial \rho}{\partial v} + \frac{\partial \log \sqrt{\mathfrak{G}}}{\partial v} \rho\right),$$

where the accent denotes differentiation with respect to v. Comparing this with (11′), we remark that

$$V_2 = 2\int V dv + 2C,$$

where C is a constant, and consequently

$$r = -\rho + 2\int V dv + 2C. \tag{17}$$

From the formulæ (13), (14), (17) we find

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= 2\left(\int V dv + C - \rho\right)\frac{\partial X}{\partial u}, \\ \frac{\partial x}{\partial v} &= 2\left(\int V dv + C\right)\frac{\partial X}{\partial v}, \end{aligned}\right\} \tag{18}$$

and similarly in y and z.

The preceding development shows that when in these formulæ (18) we assign to ρ a particular form, they give by quadratures the family of parallel surfaces cutting the corresponding congruence orthogonally, each surface of the family being determined by a particular value of the constant C. Since ρ contains two arbitrary functions, U, V, there exists a double infinity of families of parallel surfaces whose lines of curvature in one system are represented on the sphere by a given family of great circles, and after the functions X, Y, Z have been found, the further determination of these surfaces reduces to quadratures.

Write

$$E = \Sigma\left(\frac{\partial x}{\partial u}\right)^2, \quad F = \Sigma\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}, \quad G = \Sigma\left(\frac{\partial x}{\partial v}\right)^2,$$

then from (18) we have

$$E = 4\left[\int V dv + C - \rho\right]^2 \mathfrak{E}, \quad F = 0, \quad G = 4\left[\int V dv + C\right]^2. \tag{19}$$

From the last of these expressions, it follows that G is a function of v alone; hence the lines of curvature whose spherical representation is a family of great circles, are geodesics on the surface, that is, they are plane curves.

4. Recalling the Rodriques formulæ,*

$$\frac{\partial x}{\partial u} = \rho_1 \frac{\partial X}{\partial u}, \quad \frac{\partial x}{\partial v} = \rho_2 \frac{\partial X}{\partial v},$$

where ρ_1 and ρ_2 are the principal radii of curvature of the surface, and comparing them with (18), we get

$$\rho_1 = 2\left(\int V dv + C - \rho\right), \quad \rho_2 = 2\left(\int V dv + C\right). \tag{20}$$

From the second we have that ρ_2 is a function of v alone.

Conversely, let Σ be a surface for which ρ_2 is a function of v alone; we may write

$$\rho_2 = 2\left(\int V dv + C\right).$$

Denote by 2ρ the distance between the focal points and by r the distance from the middle point to the surface; then

$$\rho_2 = (r + \rho) = 2\,(V dv + C).$$

From this we have

$$\frac{\partial r}{\partial u} = -\frac{\partial \rho}{\partial u}, \quad \frac{\partial r}{\partial v} = -\frac{\partial \rho}{\partial v} + 2V,$$

and if x_1, y_1, z_1 are the coordinates of the middle point,

$$\Sigma X \frac{\partial x_1}{\partial u} = \frac{\partial \rho}{\partial u}, \quad \Sigma X \frac{\partial x_1}{\partial v} = \frac{\partial \rho}{\partial v} - 2V.$$

For any normal congruence referred to its developables and with x_1, y_1, z_1 for coordinates of the middle point, we have†

$$\Sigma X \frac{\partial x_1}{\partial u} = \quad \frac{\partial \rho}{\partial u} + \frac{\partial \log \mathcal{G}}{\partial u}\rho,$$

$$\Sigma X \frac{\partial x_1}{\partial v} = -\frac{\partial \rho}{\partial v} - \frac{\partial \log \mathfrak{E}}{\partial v}\rho.$$

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* Bianchi, Lezioni, p. 101.

† Bianchi, Lezioni, p. 262.

Comparing these with the preceding, we have

$$\rho \frac{\partial \log \mathcal{G}}{\partial u} = 0, \quad \frac{\partial \rho}{\partial v} + \rho \frac{\partial \log \sqrt{\mathfrak{G}}}{\partial v} = V.$$

These equations are satisfied by $\rho = 0$, $V = 0$. In this case $\rho_1 = \rho_2$, and hence Σ is a sphere. Since every line on a sphere is a line of curvature, it follows that the sphere is an evident solution of our problem. However, when Σ is not a sphere, the first equation shows that $\mathcal{G}$ is a function of v alone, and the second equation which ρ must satisfy is the equation (11′). Hence, *for all surfaces having one of their principal radii constant along each line of curvature in one system, the spherical representation of the other lines of curvature is a family of great circles and these lines are geodesics on the surface.*

5. Consider now the surfaces for which the lines of curvature $u =$ const. are geodesics ; then G is a function of v alone. In this case the second Codazzi equation* reduces to

$$\frac{\partial D''}{\partial u} = 0,$$

from which it follows that D'' is a function of v alone. Hence ρ_2, which is equal to $-\dfrac{D''}{G}$, is a function of v alone, and consequently the surfaces which we are discussing are characterized by the property that their lines of curvature in one system are geodesics.

The first and second sheets of the evolute of any surface may be defined as the envelopes of the planes through a point on the surface and perpendicular to the tangents to the lines of curvature $v =$ const., $u =$ const. through the point. For the surfaces which we are considering, the former plane is the same at all points along a curve $u =$ const. and is, in fact, the plane of the curve. Hence this plane depends entirely upon one parameter u, and, consequently, its envelope is a developable surface. Conversely, when the first sheet is a developable, the curves $u =$ const., on the surface are geodesics. From this it is seen that the surfaces which we have been considering are the very ones which Monge dis-

* Ib., p. 91.

cussed as surfaces for which one of the focal sheets is a developable surface and which are known as the *surfaces of Monge.** Hence, *surfaces whose lines of curvature in one system are represented on the sphere by great circles are surfaces of Monge, and conversely.*

6. Monge has shown that the most general surfaces of this kind are generated by an invariable plane curve whose plane rolls without sliding upon a developable surface. The radius of curvature of the curve is, in this case, the radius of principal curvature of the surface corresponding to the lines of curvature which are the successive positions of the plane curve, and consequently the latter radius depends only upon the parameter of these lines of curvature; in the preceding development this radius was denoted by ρ_2. We have found for its expression,

$$\rho_2 = 2\int V\,dv + 2C. \tag{21}$$

Since v is the parameter of the curvature, it follows from this expression that for a definite form of the function V the character of the curve is completely determined and the variation of C gives parallel curves with the same evolute. Recalling the expression for the semi-focal distance,

$$\rho = \frac{1}{\sqrt{\mathfrak{G}}}\left(\int \sqrt{\mathfrak{G}}\, V\,dv + U\right), \tag{22}$$

we remark that the arbitrary function V, which appears in this expression, is the same as in (21), and consequently when a particular form is given to the function V in (22), the character of the geodesic lines of curvature of the surface is determined, which are to be represented on the sphere by a given family of great circles. We will now find in what way the function U serves to determine the surface.

Since the planes of the great circles on the sphere are parallel to the planes enveloping the first sheet of the evolute, the intersections of the former are parallel to the lines generating the developable. Hence, when the spherical representation of the surface is given, the directions of the generatrices of the

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* Monge, "Application de l'Analyse à la Géométrie, 5[me] edition, p. 322 et seq.

developable are determined, and consequently the function U determines the manner in which these lines are arranged so as to form the developable. For example, if all the great circles on the sphere have a common diameter, all the planes of these circles intersect in this line, and consequently the generating developable of the surface is cylindrical. In this case the function U determines the character of the right section of the cylinder. The corresponding surfaces are the so-called *moulure* surfaces. We consider this special case further and show in what manner U enters into the expressions for the cartesian coordinates of the surface S.

Let the axis of z be the common diameter of all the great circles, and therefore parallel to the axis of the cylinder. The coordinates of a point on the sphere can be written

$$X = \cos u \sin v, \quad Y = \sin u \sin v, \quad Z = \cos v,$$

and from this

$$\sqrt{\mathfrak{E}} = \sin v, \quad \mathcal{G} = 1.$$

Now (22) becomes

$$\rho = \frac{1}{\sin v}\left(\int \sin v\, V dv + U\right),$$

and from (13) we have, by a quadrature, for the coordinates of the middle surface of the congruence,

$$x_1 = \int (U' \cos u + U \sin u)\, du - \cos u \int V \sin v\, dv,$$

$$y_1 = \int (U' \sin u - U \cos u)\, du - \sin v \int V \sin v\, dv,$$

$$z_1 = U \cot v - \int \frac{V \cos v \sin^2 v + \int V \sin v\, dv}{\sin^2 v}\, dv.$$

If we denote by x, y, z the coordinates of the surface S corresponding to the value zero for C in (17), and substitute the value for r and the preceding expressions for x_1, y_1, z_1 in (14), the surface S is given by

$$\left.\begin{aligned} x &= \quad 2\int U \sin u\, du + 2\cos u \int V_1 \cos v\, dv, \\ y &= -2\int U \cos u\, du + 2\sin u \int V_1 \cos v\, dv, \\ z &= -2\int V_1 \sin v\, dv, \end{aligned}\right\} \tag{23}$$

where

$$V_1 = \int V dv.$$

From the above we find for the square of the linear element of S,

$$ds^2 = 4\left(U - \int V_1 \cos v\, dv\right)^2 du^2 + V_1^2\, dv^2. \tag{24}$$

It is evident that the surfaces of revolution are a particular class of the surfaces defined by (23) and correspond to the case where the cylinder reduces to its axis. From (24) it follows that for surfaces of revolution U is a constant, and conversely.

7. Since the evolute of a line of curvature $u =$ const. is a plane curve in the same plane as the latter, and since the locus of these evolutes is the second sheet S_2 of the evolute of S, this second sheet also is a surface of Monge. Since its generating plane is the same as for S, the family of great circles on the sphere is the same for both surfaces. However, the normals to these two surfaces at corresponding points are perpendicular to one another. Consequently if the point (u, v) on the sphere is the image of a point on S, then $\left(u, v + \frac{\pi}{2}\right)$ is the image of the corresponding point on S_2. Again, since the differential dv denotes the angle between consecutive radii of the sphere along a great circle, or, what is the same thing, the angle between consecutive tangents to the geodesics $u =$ const. of the surface, the evolute of the curve

$$\rho_2 = 2V_1$$

is given by

$$\rho_2' = 2\frac{dV_1}{dv}.$$

As the generating developable is the same for both S and S_2, the function U is

the same for both. Therefore, given a surface S corresponding to a system of expressions for the functions $\sqrt{\mathfrak{E}}$, U and V_1; if, in the first $\cos v$, $\sin v$, are replaced by $-\sin v$, $\cos v$ respectively and V_1 is replaced by $\frac{dV_1}{dv}$, the corresponding surface is the second sheet of the envelope of S.

For example, if we make these changes in (23), we have the following expressions for the coordinates of the corresponding surface S_2:

$$\left.\begin{aligned} x_2 &= 2\int U \sin u\, du - 2\cos u \int \frac{dV_1}{dv} \sin v\, dv, \\ y_2 &= -2\int U \cos u\, du - 2\sin u \int \frac{dV_1}{dv} \sin v\, dv, \\ z_2 &= -2\int \frac{dV_1}{dv} \cos v\, dv. \end{aligned}\right\} \tag{25}$$

From the definition of the evolute, we have that

$$x_2 = x - \rho_2 X, \quad y_2 = y - \rho_2 Y, \quad z_2 = y - \rho_2 Z.$$

If the previously found expressions for x, y, z, ρ_2, X, Y, Z are substituted here, these equations can be brought to the form (25).

8. We have remarked that surfaces of revolution belong to the class under discussion and have given an example of a spherical representation of these surfaces. Now we wish to find all possible forms of the functions U_1, U_2, U corresponding to surfaces of revolution. We do this by expressing the condition that ρ_1 shall be a function of v alone. From (12) this gives

$$\frac{U_1 \int V \cos v\, dv + U_2 \int V \sin v\, dv + U}{U_1 \cos v + U_2 \sin v} = \Phi(v).$$

The following cases give all the possible ways in which this equation of condition is satisfied, and, furthermore, there are surfaces of revolution corresponding to each of these cases.

1°. $U_2 = 0$, $U = \lambda U_1$,

where λ is a constant equal to or different from zero. From (10) we find that

U_{12}, U_{22}, U_{32} are constants; put

$$U_{12} = c_1, \quad U_{22} = c_2, \quad U_{32} = c_3.$$

On account of the relation between $U_{11}, \ldots, U_{32}$ we can introduce two functions θ_1 and θ_2 as follows:

$$U_{11} = \sin\theta_1 \cos\theta_2, \quad U_{21} c_3 - U_{31} c_2 = \sin\theta_1 \sin\theta_2, \quad c_1 = \cos\theta_1.$$

From (10)

$$\theta_2' = U_1,$$

whence

$$\theta_2 = \gamma_1 + \int U_1\, du,$$

where γ_1 is a constant. Therefore

$$U_{11} = \sqrt{1 - c_1^2} \cos\left(\gamma_1 + \int U_1\, du\right),$$

and similarly for U_{21} and U_{31}. Hence X, Y, Z are given by quadratures and, therefore, the complete determination of the surfaces of revolution, for which $\sqrt{\mathfrak{G}} = U_1 \cos v$, reduces to quadratures. This is the case previously discussed at which time we took $U_1 = 1$.

2°. $U_1 = 0$, $U = \lambda U_2$.

This case is similar to the preceding and leads to similar results.

3°. $U_1 = \lambda U_2 = \mu U$,

where λ, μ are constants, and U is either a function of u or a constant different from zero. Recalling the preceding results and remarking that $\frac{U_1}{U_2} = \lambda$, we see that the problem of determining the direction cosines of the normals to the surface is equivalent to that of finding general helices from their intrinsic equations. By methods similar to those used in case 1°, it can be shown that this determination reduces to quadratures. Hence the surfaces of revolution, for which $\sqrt{\mathfrak{G}} = U_1(\cos v + \lambda \sin v)$, are found by quadratures.

4°. $U_1 = \lambda U_2$, $U = \mu$,

where λ, μ are constants. This case is similar to the preceding.

9. Since ρ_2 is a function of v alone, for ρ_1 to be a function of ρ_2 it also would

be a function of v alone, and, consequently, S would be a surface of revolution. Hence, *surfaces of revolution are the only surfaces of Monge which are Weingarten surfaces.*

Again, from (19),

$$\sqrt{\frac{E}{G}} = \frac{\left(\int V dv - \rho\right)\sqrt{\mathfrak{E}}}{\int V dv} = \frac{U_2\int V_1 \cos v\, dv - U_1\int V_1 \sin v\, dv - U}{V_1}.$$

For S to be an isothermic surface, it is necessary and sufficient that the numerator of the right-hand member be a product of a function of u by a function of v. It is readily seen that the four cases 1°, 2°, 3°, 4° of the preceding section are the only ones satisfying this condition; hence, *the surfaces of revolution are the only surfaces of Monge which are isothermic.*

10. It is well known that the direction cosines X, Y, Z, corresponding to a given family of great circles and their orthogonal trajectories, are particular solutions of the equation*

$$\frac{\partial^2\theta}{\partial u\,\partial v} - \frac{\partial \log\sqrt{\mathfrak{E}}}{\partial v}\frac{\partial\theta}{\partial u} = 0, \tag{26}$$

and the envelope of the plane whose equation is

$$Xx + Yy + Zz = W,$$

where W is a particular solution of the above equation, has the curves $u =$ const., $v =$ const. for lines of curvature. Moreover, the cartesian coordinates of the point of contact are†

$$x = WX + \Delta(W, X), \quad y = WY + \Delta(W, Y), \quad z = WZ + \Delta(W, Z),$$

where $\Delta(\phi, \psi)$ is the mixed differential parameter defined by

$$\Delta(\phi, \psi) = \frac{1}{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}\left\{\mathfrak{G}\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial u} - \mathfrak{F}\left(\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial v} + \frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial u}\right) + \mathfrak{E}\frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial v}\right\}.$$

The general integral of (26) is readily found to be

$$W = \cos v\int UU_1\, du + \sin v\int UU_2\, du + V, \tag{27}$$

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* Bianchi, Lezioni, p. 119. † Ib., p. 137.

where U, V are functions of u and v respectively. From (7) we see that the problem of finding X, Y, Z reduces to taking zero for V and the choice of three sets of values for U so that the resulting forms shall satisfy conditions (8) and (10). After X, Y, Z have been found and the two quadratures in (27) for any value of U have been effected, the corresponding surface S is found by differentiation. When we compare this result with those of the preceding sections, we see that the actual determination of the surfaces requires fewer quadratures, but an inspection of the above formulæ shows that the preceding method leads to a more ready geometrical interpretation of the results.

It is of interest to remark that when $V = 0$ in (27), W has the same form as $\sqrt{\mathcal{E}}$. Substituting $\sqrt{\mathcal{E}}$ for θ in (26), we find $U_1 = \lambda U_2$, where λ is a constant, taking any value. Hence, when

$$\sqrt{\mathcal{E}} = U_1(\lambda \cos v + \mu \sin v), \quad \mathcal{F} = 0, \quad \mathcal{G} = 1,$$

the envelope of the plane

$$Xx + Yy + Zz = \sqrt{\mathcal{E}}$$

is a surface of revolution having the lines $u =$ const. for meridians and $v =$ const. for parallels.

PRINCETON, *March*, 1902.

On the Invariants of a Homogeneous Quadratic Differential Equation of the Second Order.

By D. R. Curtiss.

The subject of invariants of linear differential equations under the transformation

$$\xi = \mu(x), \quad y = \lambda(x)\eta, \tag{1}$$

where $\mu(x)$ and $\lambda(x)$ are arbitrary functions of x, has been discussed by many writers. A brief summary of this work may be found in a paper by Dr. Bouton on the invariants of the general linear differential equation.* These considerations have been extended to systems of linear differential equations by Dr. Wilczynski,† at whose suggestion the work of this paper was undertaken. But although Staeckel has shown‡ that (1) is the most general point transformation which converts a homogeneous differential equation of any degree and of order greater than one into another of the same degree and order, equations of degree higher than unity have so far received very little notice.

In this paper I propose to treat the equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + 4p_2\left(\frac{dy}{dx}\right)^2 + p_4 y^2 + 4p_3 y\frac{dy}{dx} + 2q_2 y\frac{d^2y}{dx^2} + 4p_1\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 0, \tag{2}$$

where p_1, p_2, p_3, p_4, q_2 are functions of x, and determine those functions of the coefficients and of their derivatives which are the same for (2) and for any equation obtained from (2) by any transformation of form (1). A few applications will also receive brief notice.

Appell has published a paper§ in which he finds some of the invariants

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* American Journal of Mathematics, Vol. XXI, No. 2, 1899.

† Transactions of the American Mathematical Society, Vol. II, No. 1.

‡ Crelle's Journal, Vol. CXI.

§ Journal de Mathématique, 4th Series, Vol. V.

of this equation, but he investigates only so much of the subject as is useful in certain applications which form the bulk of his paper. His work will be referred to later on.

The transformations (1) evidently form an infinite continuous group which may be defined by differential equations. We may therefore use Lie's theory of such groups, and this method of treatment constitutes the main advance of this paper over Appell's on the same subject.

A.—*Seminvariants.*

Let us first transform the dependent variable alone. Functions of the coefficients and of their derivatives which remain invariant under this transformation we shall call seminvariants.

Denoting derivatives by accents, we have

$$\left.\begin{aligned} y &= \lambda(x)\,\eta, \\ y' &= \lambda(x)\,\eta' + \lambda'(x)\,\eta, \\ y'' &= \lambda(x)\,\eta'' + 2\lambda'(x)\,\eta' + \lambda''(x)\,\eta. \end{aligned}\right\} \tag{3}$$

As Appell remarks, y, y', y'' are expressed linearly in terms of η, η', η'', so that the invariants of the ternary quadratic form corresponding to (2) will be included among the seminvariants of (2), and will, in fact, be relative invariants under the general transformation, since this again transforms y, y', y'' linearly in terms of η, η', η''.

Substituting (3) in (2), we obtain the equation

$$\eta''^2 + 4\pi_2\eta'^2 + \pi_4\eta^2 + 4\pi_3\eta'\eta + 2\rho_2\eta''\eta + 4\pi_1\eta''\eta' = 0,$$

in which the coefficients have the values

$$\left.\begin{aligned} \pi_1 &= \frac{\lambda'}{\lambda} + p_1, \quad \pi_2 = \left(\frac{\lambda'}{\lambda}\right)^2 + 2p_1\frac{\lambda'}{\lambda} + p_2, \\ \rho_2 &= \frac{\lambda''}{\lambda} + 2p_1\frac{\lambda'}{\lambda} + q_2, \\ \pi_3 &= \frac{\lambda'}{\lambda}\cdot\frac{\lambda''}{\lambda} + p_1\left\{\frac{\lambda''}{\lambda} + 2\left(\frac{\lambda'}{\lambda}\right)^2\right\} + (2p_2 + q_2)\frac{\lambda'}{\lambda} + p_3, \\ \pi_4 &= \left(\frac{\lambda''}{\lambda}\right)^2 + 4p_1\frac{\lambda'}{\lambda}\cdot\frac{\lambda''}{\lambda} + 4p_2\left(\frac{\lambda'}{\lambda}\right)^2 + 2q_2\frac{\lambda''}{\lambda} + 4p_3\frac{\lambda'}{\lambda} + p_4. \end{aligned}\right\} \tag{4}$$

The infinitesimal transformations of the group (3) are obtained by putting

$$\left.\begin{aligned}\lambda(x) &= 1 + \phi(x)\,\delta t,\\ \lambda' &= \phi'\,\delta t, \text{ etc.}\end{aligned}\right\} \tag{5}$$

$\phi(x)$ being an arbitrary function of x alone, and δt an infinitesimal.

By inserting in (4) the special values (5) for λ, λ', etc., we find the infinitesimal transformations of the coefficients to be

$$\left.\begin{aligned}&\delta p_1 = \phi'\,\delta t, \quad \delta p_2 = 2p_1\phi'\,\delta t,\\ &\delta q_2 = [\phi'' + 2p_1\phi']\,\delta t,\\ &\delta p_3 = [p_1\phi'' + (2p_2 + q_2)\,\phi']\,\delta t,\\ &\delta p_4 = 2\,[q_2\phi'' + 2p_3\phi']\,\delta t.\end{aligned}\right\} \tag{6}$$

To obtain $\delta p_1'$, $\delta p_2'$, etc., $\delta p_1''$, $\delta p_2''$ etc., we need only take derivatives, of the corresponding order, of δp_1, δp_2, etc., since the independent variable is left untransformed by the group (3).

We have then, finally, the extended transformation

$$X(f)\,\delta t \equiv \sum_{i=1}^{4}\left(\frac{\partial f}{\partial p_i}\,\delta p_i + \frac{\partial f}{\partial p_i'}\,\delta p_i' + \ldots\right) + \frac{\partial f}{\partial q_2}\,\delta q_2 + \frac{\partial f}{\partial q_2'}\,\delta q_2' + \ldots \tag{7}$$

If $\qquad F(p_1, p_2, q_2, \ldots, p_1', p_2', q_2', \ldots, p_1^{(\nu)}, p_2^{(\nu)}, q_2^{(\nu)}, \ldots)$

is a seminvariant, it must be a solution of $X(f) = 0$, and conversely, any solution of this equation is a seminvariant. We shall first look for the rational integral seminvariants; we shall find that all others can be expressed in terms of these.

Let us assign to y, y', y'' the weights 0, 1, 2 respectively. Then, in order that (2) may have all of its terms of the same weight, namely 4, we must assign to p_κ the weight κ, and to q_2 the weight 2. Further, let the weight of $p_i^{(\nu)}$ be $i + \nu$, and let that of $q_2^{(\mu)}$ be $2 + \mu$, and let the weight of a product be equal to the sum of the weights of its factors. We say that an expression is isobaric of weight λ if all of its terms are of weight λ.

It is evident that there is no integral rational seminvariant of weight 1.

Seminvariants involving expressions of weight no greater than 2 must satisfy

the equation

$$\frac{\partial f}{\partial p_1}\phi' + \frac{\partial f}{\partial p_1'}\phi'' + \frac{\partial f}{\partial p_2}2p_1\phi' + \frac{\partial f}{\partial q_2}(\phi'' + 2p_1\phi') = 0\,.$$

Since ϕ is arbitrary, this breaks up into the two equations

$$\frac{\partial f}{\partial p_1} + 2p_1\frac{\partial f}{\partial p_2} + 2p_1\frac{\partial f}{\partial q_2} = 0, \quad \frac{\partial f}{\partial p_1'} + \frac{\partial f}{\partial q_2} = 0.$$

These are independent and form a complete system; hence we have two independent solutions, and these are easily found to be

$$P_2 = p_2 - p_1^2, \tag{8}$$

$$Q_2 = q_2 - p_1' - p_1^2. \tag{9}$$

Let P_3 be a seminvariant of weight 3. The most general expression isobaric of weight 3, written with undetermined constant coefficients, is

$$P_3 = ap_3 + bp_2p_1 + cq_2p_1 + dp_1^3 + ep_2' + fq_2' + gp_1'p_1 + hp_1''.$$

Hence, if this is a seminvariant, we must have

$$\begin{aligned}\frac{\delta P_3}{\delta t} = ap_1\phi'' + a\,(2p_2 + q_2)\,\phi' + bp_2\phi' + 2bp_1^2\phi' + cq_2\phi' + cp_1(\phi'' + 2p_1\phi')\\ + 3dp_1^2\phi' + 2ep_1\phi'' + 2ep_1'\phi' + f\phi''' + 2fp_1\phi'' + 2fp_1'\phi'\\ + gp_1'\phi' + gp_1\phi'' + h\phi''' = 0.\end{aligned}$$

From this condition follows the system of equations

$$\begin{aligned} &f + h = 0\,, \quad a + c + 2e + 2f + g = 0\,, \quad 2a + b = 0\,,\\ &a + c = 0\,, \quad 2b + 2c + 3d = 0\,, \quad 2e + 2f + g = 0\,.\end{aligned}$$

Of these only five are independent, so that we have three independent seminvariants of weight 3, one being

$$P_3 = p_3 - 2p_2p_1 - q_2p_1 + 2p_1^3. \tag{10}$$

The others turn out to be P_2' and Q_2'. In fact, since the independent variable is untransformed, any derivative of a seminvariant will itself be a seminvariant.

P_4 must be a solution of the four equations

$$\left.\begin{array}{l}
\dfrac{\partial f}{\partial p_1} + 2p_1\dfrac{\partial f}{\partial p_2} + 2p_1'\dfrac{\partial f}{\partial p_2'} + 2p_1''\dfrac{\partial f}{\partial p_2''} + 2p_1\dfrac{\partial f}{\partial q_2} + 2p_1'\dfrac{\partial f}{\partial q_2'} \\
\qquad + 2p_1''\dfrac{\partial f}{\partial q_2''} + (2p_2 + q_2)\dfrac{\partial f}{\partial p_3} + (2p_2' + q_2')\dfrac{\partial f}{\partial p_3'} + 4p_3\dfrac{\partial f}{\partial p_4} = 0, \\
\dfrac{\partial f}{\partial p_1'} + 2p_1\dfrac{\partial f}{\partial p_2'} + 4p_1'\dfrac{\partial f}{\partial p_2''} + \dfrac{\partial f}{\partial q_2} + 2p_1\dfrac{\partial f}{\partial q_2'} + 4p_1'\dfrac{\partial f}{\partial q_2''} + p_1\dfrac{\partial f}{\partial p_3} \\
\qquad + p_1'\dfrac{\partial f}{\partial p_3'} + (2p_2 + q_2)\dfrac{\partial f}{\partial p_3'} + 2q_2\dfrac{\partial f}{\partial p_4} = 0, \\
\dfrac{\partial f}{\partial p_1''} + 2p_1\dfrac{\partial f}{\partial p_2''} + \dfrac{\partial f}{\partial q_2'} + 2p_1\dfrac{\partial f}{\partial q_2''} = 0, \\
\dfrac{\partial f}{\partial p_1'''} + \dfrac{\partial f}{\partial q_2''} = 0.
\end{array}\right\} \quad (11)$$

Without going through the details of solving this set, we may notice that it is a complete system of four independent equations containing thirteen arguments; the nine independent solutions are P_2, P_2', P_2'', Q_2, Q_2', Q_2'', P_3, P_3' and

$$P_4 = p_4 - 4p_3p_1 + 4p_2p_1^2 + 2q_2p_1^2 - 3p_1^4 - 2q_2p_1' + p_1'^2 + 2p_1'p_1^2. \quad (12)$$

Turning now to the general case, and considering the case of irrational as well as of rational seminvariants, it is evident that any function of seminvariants alone is itself a seminvariant. The question now arises: Can we obtain a complete system of seminvariants, i. e., a set such that all other seminvariants are functionally dependent upon it? We can answer this in the affirmative; in fact, P_2, Q_2, P_3, P_4 and their successive derivatives constitute such a set. For the fundamental differential equation, taken to terms of weight ν, will contain the first ν derivatives of ϕ; we shall have then a complete system of ν equations in the $5\nu - 7$ arguments

$$\left.\begin{array}{l}
p_1, p_1', \ldots, p_1^{(\nu-1)}, \\
p_2, p_2', \ldots, p_2^{(\nu-2)}, \\
q_2, q_2', \ldots, q_2^{(\nu-2)}, \\
p_3, p_3', \ldots, p_3^{(\nu-3)}, \\
p_4, p_4', \ldots, p_4^{(\nu-4)}.
\end{array}\right\} \quad (13)$$

That these ν equations are all independent, follows at once from the fact, illustrated by (11), for the case $\nu = 4$, that each contains one and only one term of the form $\dfrac{\partial f}{\partial p_1^{(r)}}$, r being different for each equation. Accordingly, there

are $4\nu - 7$ seminvariants, and no more, functionally independent, involving the arguments (13) only. Such a set is formed by the quantities

$$\left.\begin{array}{l} P_2,\ P_2',\ \ldots\ldots,\ P_2^{(\nu-2)}, \\ Q_2,\ Q_2',\ \ldots\ldots,\ Q_2^{(\nu-2)}, \\ P_3,\ P_3',\ \ldots\ldots,\ P_3^{(\nu-3)}, \\ P_4,\ P_4',\ \ldots\ldots,\ P_4^{(\nu-4)}; \end{array}\right\} \tag{14}$$

for they involve only the arguments (13), are $4\nu - 7$ in number, and are all independent. The truth of this last statement may be shown thus: The members of any row of (14) are evidently independent of each other; the second row has in each member a term of the form $q_2^{(r)}$ not to be found in the first row; the third row has in each member a term of the form $p_3^{(r)}$ not to be found in the two preceding rows; the last row, a term of the form $p_4^{(r)}$ not to be found in any preceding row.

We have thus demonstrated

THEOREM I.—*All seminvariants, rational or otherwise, of equation* (2) *are functionally dependent on* P_2, Q_2, P_3, P_4 *and their successive derivatives.*

B.—*Invariants.*

It is evident that invariants can be functions only of the seminvariants. If we apply to (2) the transformation $\xi = \mu(x)$, we need only examine how this affects the seminvariants, obtaining in terms of them the differential equation an invariant must satisfy. It should be noted that the derivative of an invariant is not, for this general transformation, an invariant, since the independent variable is now also transformed.

We have

$$\left.\begin{aligned} \xi &= \mu(x), \\ \frac{dy}{dx} &= \frac{dy}{d\xi}\,\mu', \\ \frac{d^2y}{dx^2} &= \frac{d^2y}{d\xi^2}(\mu')^2 + \frac{dy}{d\xi}\,\mu''. \end{aligned}\right\} \tag{15}$$

Equation (2) now becomes

$$\left(\frac{d^2y}{d\xi^2}\right)^2 + 4\bar{\pi}_2\left(\frac{dy}{d\xi}\right)^2 + \bar{\pi}_4 y^2 + 4\bar{\pi}_3 y\frac{dy}{d\xi} + 2\bar{\rho}_2 y\frac{d^2y}{d\xi^2} + 4\bar{\pi}_1\frac{dy}{d\xi}\cdot\frac{d^2y}{d\xi^2} = 0,$$

where the new coefficients are

$$\bar{\pi}_1 = \frac{1}{2}\frac{\mu''}{\mu'^2} + \frac{p_1}{\mu'}, \quad \bar{\pi}_2 = \frac{1}{4}\frac{\mu''^2}{\mu'^4} + p_1\frac{\mu''}{\mu'^3} + \frac{p_2}{\mu'^2},$$

$$\bar{\rho}_2 = \frac{q_2}{\mu'^2}, \quad \bar{\pi}_3 = \frac{1}{2} q_2 \frac{\mu''}{\mu'^4} + \frac{p_3}{\mu'^3}, \quad \bar{\pi}_4 = \frac{p_4}{\mu'^4}.$$

The infinitesimal transformations are obtained by putting

$$\xi = x - \nu(x)\,\delta t, \tag{16}$$

$\nu(x)$ being an arbitrary function of x, and δt, an infinitesimal. We have written $-\nu(x)$ rather than $+\nu(x)$ so as to harmonize with the infinitesimal transformations of the dependent variable y. If we denote by δx and δy the infinitesimal transformations of x and y respectively, taken both times in the sense—new value of x or y minus the old value—we have, in our notation,

$$\delta x = -\nu(x)\,\delta t, \quad \delta y = -\phi(x)\,\delta t.$$

From (16) follows

$$\left.\begin{aligned} \mu' &= 1 - \nu'\,\delta t, \\ \mu'' &= -\nu''\,\delta t. \end{aligned}\right\} \tag{17}$$

This leads immediately to the following expressions for the infinitesimal changes in the coefficients:

$$\delta p_1 = (\nu' p_1 - \tfrac{1}{2}\nu'')\,\delta t, \quad \delta p_2 = (2\nu' p_2 - p_1\nu'')\,\delta t,$$
$$\delta q_2 = 2\nu' q_2\,\delta t, \quad \delta p_3 = (3\nu' p_3 - \tfrac{1}{2} q_2\nu'')\,\delta t, \quad \delta p_4 = 4\nu' p_4\,\delta t. \tag{17'}$$

To obtain the variation of a derivative of a function whose variation is known, we make use of the formula

$$\delta f' = \frac{d}{dx}(\delta f) - \frac{df}{dx}\cdot\frac{d}{dx}(\delta x).$$

In the present case, since $\delta x = -\nu\delta t$, this equation becomes

$$\delta f' = \frac{d}{dx}(\delta f) + \nu' f'\delta t. \tag{18}$$

In particular, we have

$$\delta p_1' = (2\nu' p_1' + \nu'' p_1 - \tfrac{1}{2}\nu''')\,\delta t.$$

The variations of the seminvariants can now be obtained. They are given

by the equations

$$\left.\begin{aligned}&\delta P_2 = 2\nu' P_2\,\delta t, \quad \delta Q_2 = (2\nu' Q_2 + \tfrac{1}{2}\nu''')\,\delta t,\\ &\delta P_3 = (3\nu' P_3 + \nu'' P_2)\,\delta t,\\ &\delta P_4 = (4\nu' P_4 + 2\nu'' P_3 + \nu''' Q_2)\,\delta t,\end{aligned}\right\} \tag{19}$$

while $\delta P_2'$, $\delta P_2''$, $\delta Q_2'$, etc., are readily calculated from (19) and (18).

Having thus applied the general infinitesimal transformation of the group to the seminvariants, we may at once write down the equation characteristic of an absolute invariant:

$$\sum_{i=2}^{4}\left(\frac{\partial f}{\partial P_i}\,\delta P_i + \frac{\partial f}{\partial P_i'}\,\delta P_i' + \ldots\right) + \frac{\partial f}{\partial Q_2}\,\delta Q_2 + \frac{\partial f}{\partial Q_2'}\,\delta Q_2' + \ldots = 0. \tag{20}$$

It is easy to verify the fact that for our present equation the following statements, quoted almost verbatim from the paper of Dr. Wilczynski, already referred to, hold true equally as well as for linear equations:

1. Every absolute invariant is isobaric in the coefficients (and therefore in the seminvariants (14)) and of weight zero.

2. An absolute invariant, rational in the seminvariants (14), must be the quotient of two relative invariants of the same weight.

3. A relative invariant is isobaric in the seminvariants (14), and if the common weight of all its terms is w, it satisfies the equation

$$\theta_w(\xi) = (\mu')^{-w}\theta_w(x), \tag{21}$$

or, for infinitesimal transformations,

$$\delta\theta_w = w\theta_w\,\nu'\delta t. \tag{22}$$

For proofs which need scarcely any alteration, Dr. Bouton's paper on the linear equation may be referred to.

The first equation of (19) shows that P_2 is a relative invariant. Therefore,

$$\theta_2 = P_2. \tag{23}$$

Clearly θ_3 must have the form $aP_2' + bQ_2' + cP_3$. Accordingly,

$$\begin{aligned}\delta\theta_3 &= \{a(3\nu' P_2' + 2\nu'' P_2) + b(3\nu' Q_2' + 2\nu'' Q_2 + \tfrac{1}{2}\nu^{(4)}) + c(3\nu' P_3 + \nu'' P_2)\}\,\delta t\\ &= 3\nu'\theta_3\,\delta t,\end{aligned}$$

from which follows $\qquad b = 0, \quad c + 2a = 0,$

so that we have $\qquad \theta_3 = P_3 - \tfrac{1}{2}P_2'. \qquad (24)$

The equation for an absolute invariant involving the seminvariants (14) up to weight 4, breaks up into the following system:

$$\left.\begin{aligned}
&\frac{\partial f}{\partial Q_2''}=0, \quad \frac{\partial f}{\partial Q_2'}=0, \\
&\tfrac{1}{2}\frac{\partial f}{\partial Q_2}+2Q_2\frac{\partial f}{\partial Q_2''}+2P_2\frac{\partial f}{\partial P_2''}+P_2\frac{\partial f}{\partial P_3'}+Q_2\frac{\partial f}{\partial P_4}=0, \\
&2Q_2\frac{\partial f}{\partial Q_2'}+2P_2\frac{\partial f}{\partial P_2'}+P_2\frac{\partial f}{\partial P_3}+5P_2'\frac{\partial f}{\partial P_2''}+5Q_2'\frac{\partial f}{\partial Q_2''} \\
&\qquad\qquad +(3P_3+P_2')\frac{\partial f}{\partial P_3'}+2P_3\frac{\partial f}{\partial P_4}=0, \\
&2P_2\frac{\partial f}{\partial P_2}+2Q_2\frac{\partial f}{\partial Q_2}+3P_2'\frac{\partial f}{\partial P_2'}+3Q_2'\frac{\partial f}{\partial Q_2'}+3P_3\frac{\partial f}{\partial P_3} \\
&\qquad\qquad +4P_2''\frac{\partial f}{\partial P_2''}+4Q_2''\frac{\partial f}{\partial Q_2''}+4P_3'\frac{\partial f}{\partial P_3'}+4P_4\frac{\partial f}{\partial P_4}=0.
\end{aligned}\right\} \tag{25}$$

These are all independent, hence there are $9-5=4$ functionally independent solutions; i. e., four absolute invariants, or five independent relative invariants. These five relative invariants are solutions of the first four equations, their left-hand members having been multiplied into $\nu^{(5)}$, $\nu^{(4)}$, $\nu'^{(3)}$, ν'' respectively in (20) (see (22)).

One of these solutions should be the discriminant of the ternary quadratic form corresponding to (2); this is

$$\bar{\theta}_6 = P_3^2 - P_2(P_4 - Q_2^2). \tag{26}$$

Two others we already know, θ_2 and θ_3. Without going through with the process of solving the equations, we know a priori another, given by Forsyth's Jacobian process:*

$$\bar{\bar{\theta}}_6 = 3\theta_2'\theta_3 - 2\theta_3'\theta_2. \tag{27}$$

For the fifth, we may take

$$\theta_6 = P_3^2 + 4P_2^3 Q_2 - 2P_2 P_3' + 2P_2' P_3. \tag{28}$$

These are all independent, as may easily be verified.

Before leaving this part of the subject, let us note an invariant θ_4 which we shall refer to later on. This is connected with the invariants already obtained

* Phil. Trans., I, 1888, pp. 407–418.

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by the relation

$$\theta_2\theta_4 - 12\theta_3^2 - 3\theta_6 + 15\bar{\theta}_6 - 2\bar{\bar{\theta}}_6 = 0.$$

Its value in terms of the seminvariants is

$$\theta_4 = 15(P_4 - Q_2^2) - 10P_3' + 2P_2'' + 12P_2Q_2. \tag{29}$$

Let us now attempt to find a complete system of independent relative invariants containing seminvariants (14) up to weight σ ($\sigma > 2$). These will be the solutions of the system of equations which (20), taken to terms of weight σ, breaks up into (excluding that one whose left-hand member is multiplied by ν' in (20)). (19) shows us that these equations are σ in number. They are linearly independent; for a consideration of (18) and (19) shows that two of them are

$$\frac{\partial f}{\partial Q_2^{(\sigma-2)}} = 0, \quad \frac{\partial f}{\partial Q_2^{(\sigma-3)}} = 0;$$

while if we write down the other equations obtained by equating to zero the coefficients of ν'', $\nu^{(3)}$, ..., in (20) in this order, each contains a term of the form $\varkappa_\rho P_2^{(\rho)} \frac{\partial f}{\partial P_2^{(\sigma-3)}}$ ($\varkappa_\rho$ a constant) which does not occur in any of the preceding equations. Hence, since the arguments are $4\sigma - 7$ in number, there are $3\sigma - 7$ functionally independent solutions.

We may now state

THEOREM II.—*The number of independent relative invariants of* (2) *containing seminvariants* (14) *of weight* σ *or less is* $3\sigma - 7$. ($\sigma > 2$).

It can now be shown that Forsyth's Jacobian process yields, when applied to the invariants already in our possession, a complete system. Let us write

$$\left.\begin{aligned}
\bar{\bar{\theta}}_6 &= 3\theta_2'\theta_3 - 2\theta_3'\theta_2,\\
\theta_9 &= 6\theta_2'\theta_6 - 2\theta_6'\theta_2,\\
\bar{\theta}_9 &= 6\theta_2'\bar{\theta}_6 - 2\bar{\theta}_6'\theta_2,\\
\bar{\bar{\theta}}_9 &= 6\theta_2'\bar{\bar{\theta}}_6 - 2\bar{\bar{\theta}}_6'\theta_2,\\
&\vdots\\
\theta_{3\lambda-6} &= (3\lambda-9)\theta_2'\theta_{3\lambda-9} - 2\theta_{3\lambda-9}'\theta_2,\\
\bar{\theta}_{3\lambda-6} &= (3\lambda-9)\theta_2'\bar{\theta}_{3\lambda-9} - 2\bar{\theta}_{3\lambda-9}'\theta_2,\\
\theta_{3\lambda-6} &= (3\lambda-9)\theta_2'\bar{\bar{\theta}}_{3\lambda-9} - 2\bar{\bar{\theta}}_{3\lambda-9}'\theta_2.\\
&\vdots
\end{aligned}\right\} \tag{30}$$

The invariants θ_2, θ_3, $\theta_{3\lambda-6}$, $\bar{\theta}_{3\lambda-6}$, $\bar{\bar{\theta}}_{3\lambda-6}$, $(\lambda = 4, 5, 6, \ldots, \sigma)$ are $3\sigma - 7$ in number, and they are functionally independent, for, taken in this order, each contains at least one seminvariant not to be found in the preceding; these are successively, P_3, P_3', P_4, P_2'', P_3'', P_4', P_2''', $\ldots$, $P_3^{(\sigma-3)}$, $P_4^{(\sigma-4)}$, $P_2^{(\sigma-2)}$. Finally, this set involves seminvariants of weight σ, and of no higher weight; for, by their law of formation, $\theta_{3\lambda-6}$, $\bar{\theta}_{3\lambda-6}$, $\bar{\bar{\theta}}_{3\lambda-6}$ contain seminvariants of weight higher by unity than any entering into $\theta_{3\lambda-9}$, $\bar{\theta}_{3\lambda-9}$, $\bar{\bar{\theta}}_{3\lambda-9}$, and of no higher weight; θ_2 contains a seminvariant of weight 2, θ_3 one of weight 3, while the highest weight of any seminvariant in θ_6, $\bar{\theta}_6$, $\bar{\bar{\theta}}_6$ is 4; a simple induction completes the proof.

We have now established

THEOREM III.—θ_2, θ_3, $\theta_{3\lambda-6}$, $\bar{\theta}_{3\lambda-6}$, $\bar{\bar{\theta}}_{3\lambda-6}$, $(\lambda = 4, 5, \ldots, \sigma)$ *form a complete system of relative invariants containing seminvariants* (14) *of weight σ or less* $(\sigma > 2)$; *all other such relative invariants are functionally dependent upon these.*

Hence, all absolute invariants, rational or irrational, containing these seminvariants, depend functionally upon the $3\sigma - 6$ independent rational absolute invariants formed from the above system of relative invariants.

Note that another system might have been obtained by substituting θ_3 for θ_2 in (30).

In passing we may notice another important class of invariant expressions. We have $d\xi = \mu'(x)\,dx$. Accordingly (see (21)), denoting by $\theta_w(\xi)$, as in (21), the invariant corresponding to θ_w formed for the transformed equation,

$$\frac{\theta_{\nu+1}(\xi)}{\theta_\nu(\xi)}\,d\xi = \frac{\mu'^{-(\nu+1)}\,\theta_{\nu+1}(x)}{\mu'^{-\nu}\,\theta_\nu(x)}\,\mu'\,dx = \frac{\theta_{\nu+1}(x)}{\theta_\nu(x)}\,dx.$$

There exists, therefore, a class of integral invariants of the form

$$R_\nu = \int \frac{\theta_{\nu+1}(x)}{\theta_\nu(x)}\,dx. \tag{31}$$

C.—*Semicanonical Form.*

We may choose λ so as to make the coefficient π_1 in the transformed differential equation vanish. Equations (4) show that this may be accomplished by putting

$$\frac{\lambda'}{\lambda} = -p_1,$$

or

$$\lambda = Ce^{-\int p_1\,dx}.$$

As the expressions for the seminvariants show, and as may be verified by substituting the above value for λ into equations (4), the transformed equation becomes

$$\eta''^2 + 4P_2\eta'^2 + P_4\eta^2 + 4P_3\eta\eta' + 2Q_2\eta\eta'' = 0. \qquad (32)$$

Its coefficients are seminvariants. This form of the differential equation, which is characterized by the condition $p_1 = 0$, will be called the semicanonical form. The transformation leading to it requires only a quadrature.

D.—*Canonical Form.*

The general transformation (1) contains two arbitrary functions, λ and μ, so that we should be able to obtain a transformed equation lacking any two terms, except of course the first. But, in general, the equations determining the values of λ and μ for such a transformation are not solvable by quadrature alone. We can, however, by mere quadratures, determine λ and μ so as to make the coefficients of yy', $y'y''$ disappear in the transformed equation; i. e., we can reduce (2) to the form,

$$\left(\frac{d^2Y}{dX^2}\right)^2 + 4I\left(\frac{dY}{dX}\right)^2 + JY^2 + 2KY\frac{d^2Y}{dX^2} = 0, \qquad (33)$$

which we shall call the canonical form.

Let the transformation reducing (2) to this form be

$$\left.\begin{aligned} y &= \Lambda(x)\,Y, \\ X &= \mathrm{M}(x). \end{aligned}\right\} \qquad (34)$$

Instead of actually carrying out such a transformation, we may determine I, J and K by means of the invariants of (2).

We find that for equation (33),

$$\theta_2 = I, \quad \theta_3 = -\tfrac{1}{2}\frac{dI}{dX} = -\tfrac{1}{2}\frac{dI}{dx}\,\frac{dx}{dX},$$
$$\theta_6 = 4I^2K, \quad \bar{\theta}_6 = -I(J - K^2).$$

Therefore, making use of (21), we have

$$\left.\begin{aligned} I &= \frac{\theta_2}{M'^2}, & \frac{-I'}{2M'} &= \frac{\theta_3}{M'^3}, \\ 4I^2K &= \frac{\theta_6}{M'^6}, & -I(J-K^2) &= \frac{\bar{\theta}_6}{M'^6}. \end{aligned}\right\} \qquad (34')$$

From these equations we can easily calculate I, J, K, their values being

$$I = \frac{\theta_2}{M'^2}, \quad J = \frac{\theta_6^2 - 16\theta_2^3\bar{\theta}_6}{16\theta_2^4 M'^4}, \quad K = \frac{\theta_6}{4\theta_2^2 M'^2}. \tag{35}$$

In addition, we obtain from (34′)

$$\left.\begin{aligned} M'' &= \left(\frac{\theta_3}{\theta_2} + \tfrac{1}{2}\frac{\theta_2'}{\theta_2}\right) M', \\ M' &= C_2\,\theta_2^{\frac{1}{2}} e^{\int_c^x \frac{\theta_3}{\theta_2} dx}, \\ M &= C_2 \int_c^x \theta_2^{\frac{1}{2}} e^{\int_c^x \frac{\theta_3}{\theta_2} dx}\, dx + C_3, \end{aligned}\right\} \tag{36}$$

C_2 and C_3 being arbitrary constants.

If we apply (34) to (2), the coefficient of $Y'Y''$ in the transformed equation will be

$$\tfrac{1}{2}\frac{M''}{M'^2} + \frac{1}{M'}\left(\frac{\Lambda'}{\Lambda} + p_1\right).$$

Since this must vanish, we have the following determination for $\Lambda(x)$:

$$\Lambda(x) = C_1\,\theta_2^{-\frac{1}{4}} e^{-\int_c^x \left(p_1 + \frac{1}{2}\frac{\theta_3}{\theta_2}\right) dx}. \tag{37}$$

We have, in the preceding work, taken it for granted that a transformation exists which will reduce to form (33) equation (2). We may, however, by actual substitution, readily verify the fact that (34), as determined by (36) and (37), really does reduce (2) to the desired form.

We should note one exceptional case where the canonical form fails, namely, where $\theta_2 = 0$. An extended discussion of this case will not be attempted in this paper. We may here reduce (2) to a form containing only three terms, but the transformation can no longer be obtained, in general, by mere quadratures. In fact, it may be readily verified that if ξ is any solution of the equation

$$\xi'' + Q_2\,\xi = 0,$$

the transformation

$$\left\{\begin{aligned} y &= e^{-\int_c^x p_1 dx}\,\xi\, Y, \\ X &= k\int_c^x \frac{dx}{\xi^2}, \end{aligned}\right.$$

where k is any constant, reduces (2) to the form

$$\left(\frac{d^2Y}{dX^2}\right)^2 + \frac{\xi^6\theta_3}{k^3}\,Y\frac{dY}{dX} + \frac{\xi^8(\theta_4 + 10\theta_3') + 60\theta_3\xi^7\xi'}{15k^4}\,Y^2 = 0. \tag{38}$$

At this point we may compare Appell's results. (33) is substantially his canonical form, but throughout he uses only indefinite integrals, neglecting the constants in (36) and (37); I, J, K and $\int M'\,dx$ are then absolute invariants, and these, with the derivatives of I, J, K, with regard to X, form a complete system. To show how this system may be connected with the one we have assigned, the following equations may serve as examples:

$$\frac{4K}{I} = \frac{\theta_6}{\theta_2^3}, \quad \frac{K^2 - J}{I^2} = \frac{\bar{\theta}_6}{\theta_2^3},$$

$$\frac{\left(\dfrac{dI}{dX}\right)^2}{4I^3} = \frac{\theta_3^2}{\theta_2^3}.$$

Our system has the advantage of giving us explicitly rational integral relative invariants, and also takes into account the constants of (36) and (37), thus giving the most general transformation reducing (2) to the canonical form. To find the most general transformation leaving (33) invariant, we need only substitute M'^2I for θ_2 and $-\frac{1}{2}M'^2I'$ for θ_3 in (36) and (37). The result is

$$y = C_1Y, \quad X = C_2x + C_3, \tag{39}$$

The most general infinitesimal transformation leaving the semicanonical form invariant is easily deduced. The total infinitesimal change in a coefficient for transformations (5) and (16) together is evidently the sum of the changes due to the transformation of each variable separately. We wish, then, to obtain all those transformations which make δp_1 vanish if p_1 is zero. From (6) and (17') we find

$$\delta p_1 = (\phi' + \nu' p_1 - \tfrac{1}{2}\nu'')\,\delta t.$$

Hence for any infinitesimal transformation leaving the semicanonical form unchanged,

$$\phi' - \tfrac{1}{2}\nu'' = 0; \text{ i. e., } \phi = \tfrac{1}{2}\nu' + c \qquad (c \text{ an arbitrary constant}).$$

This gives the infinite subgroup, whose infinitesimal transformations are

$$\left.\begin{aligned}\delta y &= -\phi(x)\,y\,\delta t = -\left(\tfrac{1}{2}\nu'(x)+c\right)y\,\delta t,\\ \delta x &= -\nu(x)\,\delta t.\end{aligned}\right\} \tag{40}$$

The corresponding finite transformations are

$$\left.\begin{aligned}y &= k\mu'^{-\frac{1}{2}}\eta \qquad (k \text{ an arbitrary constant}),\\ \xi &= \mu(x).\end{aligned}\right\} \tag{41}$$

The equations for the most general subgroup leaving the canonical form invariant are

$$\phi' - \tfrac{1}{2}\nu'' = 0, \quad (2p_2+q_2)\phi' - \tfrac{1}{2}q_2\nu'' = 0,$$

giving

$$\phi = k_1, \quad \nu = k_2x + k_3.$$

The infinitesimal transformations sought for are therefore

$$\left.\begin{aligned}\delta y &= -k_1 y\,\delta t,\\ \delta x &= -(k_2x+k_3)\,\delta t,\end{aligned}\right\} \tag{42}$$

and, of course, (39) gives the corresponding finite transformations. We should note that under the subgroup (39) the derivative of a relative invariant is itself invariant. For, by (18) and (22),

$$\delta\theta'_\sigma = \frac{d}{dx}(\delta\theta_\sigma) + \nu'\theta'_\sigma\delta t = \{\sigma\nu''\theta_\sigma + (\sigma+1)\nu'\theta'_\sigma\}\,\delta t.$$

But, since for all infinitesimal transformations of the subgroup $\nu''=0$,

$$\delta\theta'_\sigma = (\sigma+1)\,\nu'\,\theta'_\sigma\,\delta t;$$

i. e., θ'_σ is an invariant of the subgroup. Clearly any transformation (2) agreeing with (39) in its change of independent variable has the same property.

E.—*Remarks and Applications.*

If θ_2 vanishes identically, the complete system of Theorem III apparently reduces to θ_3 alone, while if both θ_2 and θ_3 vanish, every member of the system become equal to zero. The vanishing of θ_2 and θ_3 means no more or less than the vanishing of P_2 and P_3; hence we might go back to the differential equation (20), omitting terms in P_2, P_3 and their derivatives, and from its solutions build up a new complete system. Nevertheless, the system of Theorem III, though all its members vanish, is still complete; all relative invariants may be obtained

as the *limits* of combinations of its members; an illustration is afforded by θ_4 which equals $15(P_4 - Q_3^2)$ when $\theta_2 = \theta_3 = 0$.

For some purposes a form of (2), in which $(y'')^2$ has a coefficient p_0 different from unity, may be advantageous; it will be general enough to take p_0 as a constant. In this case p_0 is absolutely invariant. Seminvariants and relative invariants are readily obtained from those given in this paper by writing $\frac{p_1}{p_0}$, $\frac{p_2}{p_0}$, etc., for p_1, p_2, etc., and multiplying the resulting forms by a power of p_0 sufficient to clear of fractions. Isobaric seminvariants and invariants are then homogeneous in the coefficients. From these results we could easily construct the invariants of a form of (2) lacking the term in $(y'')^2$.

In conclusion, a few applications are here given, the discussion in each case being as brief as is consistent with clearness.

1. Two equations of form (2) are equivalent under a transformation (3) of the dependent variable alone, when all the seminvariants of the one are the same functions of the independent variable as are those of the other. This condition is necessary, as is clear from the definition of a seminvariant, and it is also sufficient, since the semicanonical forms of the two equations are then identical.

2. Two equations of form (2) for which $\theta_2 \neq 0$ are equivalent under a transformation of the group (1) if a transformation $\xi = \bar{\mu}(x)$ exists which, when applied to the absolute invariants entering into the canonical form of one equation, gives the corresponding invariants of the other equation.

First, this is necessary; for, suppose a transformation $\xi = \bar{\mu}(x)$, $y = \bar{\lambda}(x)\eta$ changes an equation (A) in terms of x and y into an equation (B) in terms of ξ and η. The absolute invariants of (B) are in terms of ξ, and, from the nature of an invariant, must *all* be identical with the corresponding invariants of (A) if $\bar{\mu}(x)$ be substituted for ξ.

This condition is also sufficient; for transform (A) into an equation (C) in ξ and y by means of the substitution $\xi = \bar{\mu}(x)$. The invariants of (C) are now identical with those of (B); and we may easily show that (C) and (B) are equivalent under a transformation $y = \bar{\lambda}(x)\eta = \lambda(\xi)\eta$ of the dependent variable alone. To prove this, reduce both to the canonical form. Now $\frac{\theta_3}{\theta_2}\,d\xi$ and $\theta_2^{\frac{1}{2}}\,d\xi$ are both absolute invariants; this may be shown as in the work preceding (31); hence, choosing C_2, C_3, and c the same for each (see (36)), M is identically the same

for (C) as for (B). I, J, K are the products of absolute invariants and expressions of the form

$$\left(\frac{1}{C_2 e^{\int_c^\xi \frac{\theta_3}{\theta_2} d\xi}}\right)^r,$$

r being either 2 or 4, and are therefore also the same for (C) and (B). Thus (C) and (B) are reduced to identically the same equation by transformations which agree so far as the independent variable is concerned ; they are equivalent under a transformation of the dependent variable alone ; (A) is transformed into (B) by the substitution of variables $\xi = \bar{\mu}(x)$, $y = \bar{\lambda}(x)\eta$.

To apply this test to two equations in x and ξ, we should equate the absolute invariants entering into the canonical form of each; if these relations all give the same solution for ξ in terms of x, the two equations are equivalent. In particular, if one has constant coefficients, the absolute invariants of the other must reduce to constants.

Note that the preceding work shows that if the absolute invariants entering into the canonical form of the one are carried into the corresponding invariants of the other by the transformation $\xi = \bar{\mu}(x)$, the equations are equivalent, and, therefore, this same transformation carries *all* the absolute invariants of the one into those of the other. Thus the invariants of an equation (2) are completely determined by those which enter into the canonical form.

Remembering that one invariant entering into the canonical form is $\frac{\theta_3}{\theta_2}\,dx$, it will at once be seen that the equivalence condition may be given in the following form:

$$\theta_2(\xi) = \left(\frac{d\xi}{dx}\right)^{-2}\theta_2(x), \quad \theta_3(\xi) = \left(\frac{d\xi}{dx}\right)^{-3}\theta_3(x),$$
$$\theta_6(\xi) = \left(\frac{d\xi}{dx}\right)^{-6}\theta_6(x), \quad \bar{\theta}_6(\xi) = \left(\frac{d\xi}{dx}\right)^{-6}\bar{\theta}_6(x),$$

the invariants on the left-hand side of each equation being formed for the equation in ξ, those on the right for the equation in x.

We have assumed here that $\theta_2 \neq 0$, i. e., that a canonical form is possible; we shall not in this paper attempt the discussion of the case $\theta_2 = 0$.

3. If $\bar{\theta}_6$ vanishes identically, (2) breaks up into two linear equations of the second order.

4. If $\theta_2 = \theta_3 = \theta_4 = 0$, (2) is the square of a linear equation of the second order; (2) then has no invariants. For if $\theta_2 = \theta_3 = \theta_4 = 0$,

$$P_2 = P_3 = 0,$$
$$P_4 - Q_2^2 = 0;$$

the form (38) reduces to

$$(Y'')^2 = 0.$$

Form (38) also gives us a binomial form

$$\left(\frac{d^2 Y}{dX^2}\right)^2 + \frac{\xi^8 \theta_4}{15 K^4} Y^2 = 0,$$

to which (2) is reducible in case $\theta_2 = \theta_3 = 0$.

5. Appell, in the article already referred to, has developed the condition that (2) should have for its general solution

$$y = h^2 u_1 + hk u_2 + k^2 u_3,$$

h and k being arbitrary constants, and u_1, u_2, u_3 linearly independent solutions. The conditions he develops are $K = C$, $J = C^2 + 4CI$, C being a constant. One of these conditions may be replaced by the relation $\theta_6 + \bar{\theta}_6 = 0$.

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Surfaces of Constant Mean Curvature.

By L. P. Eisenhart.

Cosserat has established the following theorem:*

Let ϕ be any function of two variables u and v; if z denotes any solution of the equation

$$\frac{\partial^2 z}{\partial u \partial v} - \frac{1}{2} \frac{\dfrac{\partial^2 \phi}{\partial u \partial v}}{\dfrac{\partial \phi}{\partial u}} \frac{\partial z}{\partial u} - \frac{1}{2} \frac{\dfrac{\partial^2 \phi}{\partial u \partial v}}{\dfrac{\partial \phi}{\partial v}} \frac{\partial z}{\partial v} = 0, \tag{1}$$

the formulæ

$$x + iy = \phi, \quad x - iy = -\int \frac{\left(\dfrac{\partial z}{\partial u}\right)^2}{\dfrac{\partial \phi}{\partial u}} du + \frac{\left(\dfrac{\partial z}{\partial v}\right)^2}{\dfrac{\partial \phi}{\partial v}} dv, \qquad z = z, \tag{2}$$

give the cartesian coordinates x, y, z of a surface for which the parametric lines are of length zero; and the square of the linear element is

$$ds^2 = - \frac{\left(\dfrac{\partial z}{\partial u} \dfrac{\partial \phi}{\partial v} - \dfrac{\partial z}{\partial v} \dfrac{\partial \phi}{\partial u}\right)^2}{\dfrac{\partial \phi}{\partial u} \dfrac{\partial \phi}{\partial v}} du dv. \tag{3}$$

This method for the determination of a surface lends itself particularly to surfaces of constant mean curvature. We find that for this class of surfaces the function ϕ satisfies a partial differential equation of the fourth order, and that when a particular integral is found, the further determination of the corresponding surface requires quadratures only. Several particular integrals can be found; in some of these cases the surface is imaginary.

* Comptes Rendus, 125, pp. 159–162.

From (3) we have

$$E=\Sigma\left(\frac{\partial x}{\partial u}\right)^2=0, \quad F=\Sigma\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}=-\tfrac{1}{2}\,\frac{\left(\dfrac{\partial z}{\partial u}\,\dfrac{\partial \phi}{\partial v}-\dfrac{\partial z}{\partial v}\,\dfrac{\partial \phi}{\partial u}\right)^2}{\dfrac{\partial \phi}{\partial u}\,\dfrac{\partial \phi}{\partial v}},$$

$$G=\Sigma\left(\frac{\partial x}{\partial v}\right)^2=0. \tag{4}$$

Differentiating the first of these expressions with respect to u, we have

$$\Sigma\frac{\partial x}{\partial u}\,\frac{\partial^2 x}{\partial u^2}=0,$$

Associate with this the identity

$$\Sigma X\frac{\partial x}{\partial u}=0,$$

where X, Y, Z denote the direction cosines of the normal to the surface, then we have the relations

$$\frac{\dfrac{\partial x}{\partial u}}{Y\dfrac{\partial^2 z}{\partial u^2}-Z\dfrac{\partial^2 y}{\partial u^2}}=\frac{\dfrac{\partial y}{\partial u}}{Z\dfrac{\partial^2 x}{\partial u^2}-X\dfrac{\partial^2 z}{\partial u^2}}=\frac{\dfrac{\partial z}{\partial u}}{X\dfrac{\partial^2 y}{\partial u^2}-Y\dfrac{\partial^2 z}{\partial u^2}}.$$

In consequence of (4) we have

$$\Sigma\left(Y\frac{\partial^2 z}{\partial u^2}-Z\frac{\partial^2 y}{\partial u^2}\right)^2=\Sigma X^2.\Sigma\left(\frac{\partial^2 x}{\partial u^2}\right)^2-\left(\Sigma X\frac{\partial^2 x}{\partial u^2}\right)^2=0,$$

and hence if we adopt the notation

$$D=\Sigma X\frac{\partial^2 x}{\partial u^2}, \quad D'=\Sigma X\frac{\partial^2 x}{\partial u\,\partial v}, \quad D''=\Sigma X\frac{\partial^2 x}{\partial v^2},$$

the above identity gives

$$D^2=\Sigma\left(\frac{\partial^2 x}{\partial u^2}\right)^2; \tag{5}$$

in like manner it can be shown that

$$D'^2=\Sigma\left(\frac{\partial^2 x}{\partial u\,\partial v}\right)^2, \quad D''^2=\Sigma\left(\frac{\partial^2 x}{\partial v^2}\right)^2. \tag{6}$$

Substituting for x, y, z their expressions from (2) and making use of (1), we get

$$D^2 = \frac{\left(\frac{\partial^2\phi}{\partial u^2}\frac{\partial z}{\partial u} - \frac{\partial\phi}{\partial u}\frac{\partial^2 z}{\partial u^2}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2}, \quad D''^2 = \frac{\left(\frac{\partial^2\phi}{\partial v^2}\frac{\partial z}{\partial v} - \frac{\partial\phi}{\partial v}\frac{\partial^2 z}{\partial v^2}\right)^2}{\left(\frac{\partial\phi}{\partial v}\right)^2},$$

$$D'^2 = \tfrac{1}{4}\left(\frac{\partial z}{\partial u}\frac{\partial\phi}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial\phi}{\partial u}\right)^2 \frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial\phi}{\partial v}\right)^2}, \tag{7}$$

The expression for the mean curvature takes the simple form*

$$\tfrac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = -\frac{D'}{F}, \tag{8}$$

where ρ_1 and ρ_2 are the principal radii of curvature; hence for surfaces of constant mean curvature $\varkappa$, we have

$$\frac{D'}{F} = -\varkappa.$$

Substituting D' and F their expressions from (7) and (4), we have

$$\frac{\partial^2\phi}{\partial u\,\partial v} = \pm\,\varkappa\left(\frac{\partial z}{\partial u}\frac{\partial\phi}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial\phi}{\partial u}\right).$$

Since the sign of the mean curvature of a surface is determined by the assigned positive direction of the normal, it is evident that there is no loss of generality if we write the above expression as follows:

$$\frac{\partial^2\phi}{\partial u\,\partial v} = \varkappa\left(\frac{\partial z}{\partial u}\frac{\partial\phi}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial\phi}{\partial u}\right). \tag{9}$$

By retracing the steps in the above development, we find that the converse of this result is true. Hence, the necessary and sufficient condition that formulæ (2) define a surface of constant mean curvature $\varkappa$ is that ϕ and z satisfy equations (1) and (9) simultaneously.

When $\varkappa = 0$, that is, when the surface is minimal, equations (9) and (1) reduce to

$$\frac{\partial^2\phi}{\partial u\,\partial v} = 0, \quad \frac{\partial^2 z}{\partial u\,\partial v} = 0,$$

* Bianchi, Lezioni, p. 104.

hence

$$\phi = U_1 + V_1, \quad z = U_2 + V_2,$$

where U_1 and U_2 are functions of u alone, V_1 and V_2 are functions of v alone. Substituting these expressions for ϕ and z in (2), we find the well-known property of minimal surfaces, namely, that their cartesian coordinates are expressible as a sum of two functions, each of one of the parameters of its lines of length zero. That this is a characteristic property can be seen from (9). For, in order that $\frac{\partial^2\phi}{\partial u\,\partial v}$ be zero, it is necessary that $\varkappa$ be zero; otherwise

$$\frac{\partial z}{\partial u}\frac{\partial \phi}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial \phi}{\partial u} = 0,$$

that is, z is expressible as a function of ϕ and likewise x and y, and consequently the formulæ (2) would define a curve and not a surface. We will not discuss any further the case where $\varkappa = 0$, but in what follows it will be understood that $\varkappa \neq 0$.

From (9) we see that if ϕ is a function of u alone or a function of v alone, the above equation holds, and consequently the surface reduces to a curve. Hence ϕ is a function of both u and v.

Eliminating z between equations (1) and (9), we find that for equations (2) to define a surface of constant mean curvature $\varkappa$, the function ϕ must satisfy the following partial differential equation of the fourth order:

$$\frac{\partial^4\phi}{\partial u^2\partial v^2} = \left(\frac{\frac{\partial^2\phi}{\partial u^2}}{\frac{\partial\phi}{\partial u}} + \frac{\frac{\partial^2\phi}{\partial u\,\partial v}}{\frac{\partial\phi}{\partial v}}\right)\frac{\partial^3\phi}{\partial u\,\partial v^2} + \left(\frac{\frac{\partial^2\phi}{\partial u\,\partial v}}{\frac{\partial\phi}{\partial u}} + \frac{\frac{\partial^2\phi}{\partial v^2}}{\frac{\partial\phi}{\partial v}}\right)\frac{\partial^3\phi}{\partial u^2\,\partial v}$$

$$- \frac{\frac{\partial^2\phi}{\partial u^2}\,\frac{\partial^2\phi}{\partial u\,\partial v}\,\frac{\partial^2\phi}{\partial v^2}}{\frac{\partial\phi}{\partial u}\,\frac{\partial\phi}{\partial v}} - \left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2\left(\frac{\frac{\partial^2\phi}{\partial u^2}}{\left(\frac{\partial\phi}{\partial u}\right)^2} + \frac{\frac{\partial^2\phi}{\partial v^2}}{\left(\frac{\partial\phi}{\partial v}\right)^2}\right). \quad (10)$$

It is of interest to note that this equation does not involve $\varkappa$, and hence the general solution leads to surfaces of any constant mean curvature whatever.

Solve equation (9) for $\frac{\partial z}{\partial v}$ and substitute its expression in (1); then, by a

quadrature, we find

$$\frac{\partial z}{\partial u}=\frac{1}{2\varkappa}\,\frac{\partial\phi}{\partial u}\left(-\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2\frac{\partial\phi}{\partial v}}\,dv+U\right), \tag{11}$$

where U is a function of u alone, whose form is perfectly determinate for a value of ϕ, as we shall see in a moment. In a similar way we find

$$\frac{\partial z}{\partial v}=\frac{1}{2\varkappa}\,\frac{\partial\phi}{\partial v}\left(\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\frac{\partial\phi}{\partial u}\left(\frac{\partial\phi}{\partial v}\right)^2}\,du+V\right), \tag{12}$$

where V is a determinate function of v alone. Since $\frac{\partial^2\phi}{\partial u\,\partial v}\neq 0$, the condition of integrability of the expressions (11) and (12) reduces to

$$\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\frac{\partial\phi}{\partial u}\left(\frac{\partial\phi}{\partial v}\right)^2}\,du+\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2\frac{\partial\phi}{\partial v}}\,dv+2\,\frac{\frac{\partial^2\phi}{\partial u\,\partial v}}{\frac{\partial\phi}{\partial u}\,\frac{\partial\phi}{\partial v}}+V-U=0. \tag{13}$$

If this equation be differentiated with respect to u and v, the resulting equation can be brought to the form (10). Hence, for every particular integral ϕ of (10), the sum of the first three terms of (13) reduces to the sum of a function of u alone and a function of v alone. Thus, given an integral ϕ_1, we find that the sum of the first three terms reduces to U_1+V_1, where U_1 and V_1 are readily found. Then from (13) we have

$$U=U_1+c,\quad V=-V_1+c,$$

and, consequently, U and V are known. Therefore, having found an integral of (10) and the corresponding function U and V by means of (13), we get the z coordinate of the corresponding surface of mean curvature $\varkappa$ by the quadrature

$$z=\frac{1}{2\varkappa}\int\left[\frac{\partial\phi}{\partial u}\left(-\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2\frac{\partial\phi}{\partial v}}\,dv+U\right)du\right.$$
$$\left.+\frac{\partial\phi}{\partial v}\left(\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\frac{\partial\phi}{\partial u}\left(\frac{\partial\phi}{\partial v}\right)^2}\,du+V\right)dv\right]. \tag{14}$$

From the form of equation (10) we see that an integral is determined only to within a constant factor, and we shall find that this factor fixes the magnitude of the mean curvature. Consider an integral ϕ of (10) which does not involve a constant factor, and in (2) and (14) replace ϕ by $c\phi$, where c is a constant. In place of U and V we must put U/c and V/c in consequence of (13). From this we see that the quantity under the integral sign does not vary with c. By means of (14) the expression for $x-iy$ can be put in the form $\frac{1}{c\varkappa^2}\Phi$, where Φ is a function independent of c and $\varkappa$. Consider now two surfaces, of coordinates x_1, y_1, z_1 and x_2, y_2, z_2 corresponding to the same function ϕ and the values c_1 and c_2 of the constant c. It is evident that the mean curvatures of the two surfaces are unequal; call them $\varkappa_1$ and $\varkappa_2$. Then we get from (2)

$$\frac{x_1+iy_1}{c_1}=\frac{x_2+iy_2}{c_2}, \quad c_1\varkappa_1^2(x_1-iy_1)=c_2\varkappa_2^2(x_2-iy_2),$$

from which it follows that

$$c_1\varkappa_1=c_2\varkappa_2.$$

Hence, by choosing a suitable unit we can put $c=\frac{1}{\varkappa}$. We have then the following theorem:

Given an integral ϕ *of equation* (10) *which does not involve a constant factor; the formulæ*

$$\varkappa+iy=\phi/\varkappa, \quad x-iy=-\varkappa\int\frac{\left(\frac{\partial z}{\partial u}\right)^2}{\frac{\partial\phi}{\partial u}}du+\frac{\left(\frac{\partial z}{\partial v}\right)^2}{\frac{\partial\phi}{\partial v}}dv, \tag{15}$$

and (14) *define a surface of constant mean curvature* $\varkappa$.

However, all surfaces defined by these formulæ and corresponding to the same integral ϕ are homothetic to one another, and consequently, for the further discussion, there will be no lack of generality if we put $\varkappa=1$ in (15).

Suppose we put $u=U_2$ and $v=V_2$, where U_2 and V_2 are any functions whatever of new parameters u_1 and v_1 respectively, defined in this manner, and denote by ϕ_1 the result of replacing u and v in ϕ by u_1 and v_1 respectively. It is evident that if ϕ is an integral of equation (10), ϕ_1 is an integral of the equation

obtained from (10) by replacing u and v by u_1 and v_1; we shall refer to this equation as (10′). From the above we have

$$\frac{\partial\phi_1}{\partial u_1}=\frac{\partial\phi_1}{\partial u}U_2', \quad \frac{\partial\phi_1}{\partial v_1}=\frac{\partial\phi_1}{\partial v}V_2', \ldots,$$

where the primes denote differentiation. If these values for $\frac{\partial\phi_1}{\partial u_1}\ldots\frac{\partial^4\phi_1}{\partial u_1^2\,\partial v_1^2}$ are substituted in equation (10′), we find, after some easy reductions,

$$\frac{\partial^4\phi_1}{\partial u^2\,\partial v^2}=\left(\frac{\dfrac{\partial^2\phi_1}{\partial u^2}}{\dfrac{\partial\phi_1}{\partial u}}+\frac{\dfrac{\partial^2\phi_1}{\partial u\,\partial v}}{\dfrac{\partial\phi_1}{\partial v}}\right)\frac{\partial^3\phi_1}{\partial u\,\partial v^2}+\left(\frac{\dfrac{\partial^2\phi_1}{\partial u\,dv}}{\dfrac{\partial\phi_1}{\partial u}}+\frac{\dfrac{\partial^2\phi_1}{\partial v^2}}{\dfrac{\partial\phi_1}{\partial v}}\right)\frac{\partial^3\phi_1}{\partial u^2\,\partial v}$$
$$-\frac{\dfrac{\partial^2\phi_1}{\partial u^2}\,\dfrac{\partial^2\phi_1}{\partial u\,\partial v}\,\dfrac{\partial^2\phi_1}{\partial v^2}}{\dfrac{\partial\phi_1}{\partial u}\,\dfrac{\partial\phi_1}{\partial v}}-\left(\frac{\partial^2\phi_1}{\partial u\,\partial v}\right)^2\left(\frac{\dfrac{\partial^2\phi_1}{\partial u^2}}{\left(\dfrac{\partial\phi_1}{\partial u}\right)^2}+\frac{\dfrac{\partial^2\phi_1}{\partial v^2}}{\left(\dfrac{\partial\phi_1}{\partial v}\right)^2}\right).$$

Comparing this with (10), we have the result: *Given any function $\phi(u, v)$ satisfying equation* (10), *then $\phi(U, V)$ is an integral, where U and V any functions of u and v, respectively.*

In the same way it can be shown that equation (13) takes the form

$$\int\frac{\left(\dfrac{\partial^2\phi_1}{\partial u\,\partial v}\right)^2}{\dfrac{\partial\phi_1}{\partial u}\left(\dfrac{\partial\phi_1}{\partial v}\right)^2}du+\int\frac{\left(\dfrac{\partial^2\phi_1}{\partial u\,\partial v}\right)^2}{\left(\dfrac{\partial\phi_1}{\partial u}\right)^2\dfrac{\partial\phi_1}{\partial u}}dv+2\frac{\dfrac{\partial^2\phi_1}{\partial u\,\partial v}}{\dfrac{\partial\phi_1}{\partial u}\,\dfrac{\partial\phi_1}{\partial v}}+V_1-U_1=0.$$

Hence, the functions U_1 and V_1 corresponding to the solution ϕ_1 of equation (10) are gotten from those corresponding to the solution ϕ by replacing u and v by U and V. In a similar manner it can be shown that the expressions for $x-iy$ and z corresponding to the solution ϕ_1 are gotten from the corresponding ones for the solution ϕ by the same substitution. However, since a change of parameters doesn't affect the surface, and since the same lines are parametric when u is replaced by a function of u and v by a function of v, it is evident that all functions ϕ which can be brought to the same form by such a change determine the same surface.

We shall consider now the surfaces corresponding to several evident integrals of equation (10) and several which are readily found indirectly.

A particular solution of equation (10) is given by

$$\phi = uv. \tag{16}$$

When this expression for ϕ is substituted in (13), the latter reduces to

$$U - V = 0,$$

hence

$$U = V = \alpha,$$

where α is a constant. Putting these values for ϕ, U and V in (14) and (15), we get

$$x + iy = uv, \quad x - iy = \tfrac{1}{4}\left(\frac{1}{uv} - 2\alpha \log \frac{u}{v} - \alpha^2 uv\right),$$

$$z = \tfrac{1}{2}\left(\log \frac{u}{v} + \alpha uv\right). \tag{17}$$

Eliminating u and v, we get the following equation of the surface

$$4(x^2 + y^2 + z^2) = 1 + [2z - \alpha(x + iy)]^2, \tag{18}$$

Hence, unless α is zero, the surface is imaginary. When $\alpha = 0$, equation (18) reduces to

$$x^2 + y^2 = \tfrac{1}{4},$$

that is, the surface is a right circular cylinder. Recalling the preceding results, we have that *the cylinder of revolution is the only real surface of constant mean curvature corresponding to the function* $\phi = UV$, *where* U *and* V *are any functions whatever of* u *and* v *respectively.*

Another evident integral of equation (10) is

$$\phi = \frac{1}{u + v}. \tag{19}$$

As in the preceding case, equation (13) for this expression of ϕ reduces to

$$U - V = 0,$$

hence,

$$U = V = \alpha.$$

When these expressions for ϕ, U and V are substituted in (14) and (15), they give

$$x + iy = \frac{1}{u + v}, \quad x - iy = \frac{4uv + 2\alpha(u - v) - \alpha^2}{u + v}, \quad z = \frac{v - u + \alpha}{u + v}. \tag{20}$$

If these values for ϕ and z are put in the expressions (7) for D and D'', we get

$$D = D'' = 0,$$

that is, the lines of length zero are asymptotic lines for the surface. Hence S is a sphere.* In fact, this is readily seen by eliminating u and v from the expressions (20). If we introduce parameters u_1 and v_1 defined by

$$u_1 = 2u - \alpha, \quad v_1 = 2v + \alpha,$$

the expressions (20) become

$$x + iy = \frac{2}{u_1 + v_1}, \quad x - iy = \frac{2u_1 v_1}{u_1 + v_1}, \quad z = \frac{v_1 - u_1}{u_1 + v_1}, \tag{20'}$$

and by elimination

$$x^2 + y^2 + z^2 = 1.$$

Hence, *for any function ϕ of the form $\frac{1}{U+V}$ the surface is a sphere.*

We propose now to find all integrals of equation (10) which are functions of $u + v$. If we denote by accents derivatives with respect to $u + v$, we find that equation (10) reduces to the total differential equation

$$\phi^{\text{IV}} - 4\frac{\phi''\phi'''}{\phi'} + 3\frac{\phi''^3}{\phi'^2} = 0.$$

Put $y = \phi'$, $p = \phi''$, then upon substitution and reduction the above equation becomes

$$p\left[p\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2 - 4\frac{p}{y}\frac{dp}{dy} + 3\frac{p^2}{y^2}\right] = 0.$$

If this equation is satisfied by $p = 0$, we have

$$\phi = c\,(u + v),$$

where c is a constant, which, as we have seen, is the case of minimal surfaces. We exclude this case and introduce two new functions, z and r defined by

$$p = 2y^{\frac{3}{2}} z, \quad y = e^r.$$

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* Bulletin of the Amer. Math. Soc., March, 1902, p. 241.

Then the above equation reduces to

$$z\frac{d^2z}{dr^2}+z\frac{dz}{dr}+\left(\frac{dz}{dr}\right)^2=0.$$

An integral of this equation is $z=$ const., say c; then, from the above we have

$$\phi''=2\phi'^2c,$$

whence, by two quadratures,

$$\phi=-\frac{1}{c^2(u+v)},$$

which, as we have seen, leads to the sphere.

We exclude this case now and put

$$\frac{dz}{dr}=t;$$

then the above equation becomes

$$z\frac{dt}{dz}+t+z=0,$$

which, upon integration, gives

$$t=\frac{c}{z}-\frac{z}{2}.$$

Retracing the steps in the above substitutions, we get

$$\phi=\gamma\tan\frac{u+v+\beta}{\alpha}+\delta, \tag{21}$$

where α, β, γ, δ are arbitrary constants. By a translation of the surface and a change of parameters, this expression for ϕ can be given the form

$$\phi=x+iy=\gamma\tan(u+v). \tag{21'}$$

The equation (13) reduces to

$$\frac{4(u+v)}{\gamma}+V-U=0,$$

from which we have

$$U=\frac{4u+2\delta}{\gamma},\quad V=-\frac{4v-2\delta}{\gamma},$$

where δ is a constant. From (14) and (15) we find

$$\left.\begin{aligned} z &= (u - v + \delta)\tan(u+v), \\ x - iy &= -\frac{1}{\gamma}\left[\frac{u+v}{2} - \tfrac{1}{4}\sin 2(u+v) + (u-v+\delta)^2\tan(u+v)\right] \end{aligned}\right\}. \quad (22)$$

When a suıface is referred to its lines of length zero, the directions of the lines of curvature are given by*

$$D\,du^2 - D''\,dv^2 = 0. \quad (23)$$

If the values of D and D'' for the surface given by (21′) and (22) are calculated by means of (7), we get for (23)

$$du^2 - dv^2 = 0.$$

Hence, if we put $\qquad u + v = u_1, \quad u - v = iv_1,$

u_1 and v_1 are the real parameters of the lines of curvature such that the linear element takes the form

$$ds^2 = \lambda\,(du_1^2 + dv_1^2).$$

The cartesian coordinates have the following expressions:

$$x + iy = \tan u_1, \quad x - iy = -\left[\frac{u_1}{2} - \tfrac{1}{4}\sin 2u_1 - v_1^2\tan u_1\right], \quad z = iv_1\tan u_1,$$

from which it follows that the surface is imaginary. Combining the above results with the theorem which we established just before the discussion of these particular integrals, we have the theorem:

Minimal surfaces and the sphere are the only real surfaces of constant mean curvature for which ϕ is a function of the sum of any function of u and any function of v.

Again we seek the integrals of equation (10) which are a function of uv. If we denote by accents the derivatives with respect to uv, equation (10) becomes

$$uv\,\phi'^2\phi^{IV} + 2\phi'^2\phi''' - 3\phi'\phi''^2 + 3uv\,\phi''^3 - 4uv\,\phi'\phi''\phi''' = 0.$$

Put $\phi' = e^t$, $\log uv = r$; then this equation becomes

$$\frac{d^3t}{dr^3} - \frac{dt}{dr}\,\frac{d^2t}{dr^2} - \frac{d^2t}{dr^2} = 0.$$

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* Bianchi, Lezioni, p. 99.

An integral of this equation is given by $t =$ const.; in this case

$$\phi = c\,uv,$$

which case has been considered before. Excluding this case from what follows, we make the substitution $\dfrac{dt}{dr} = \theta$, and find that θ satisfies the equation

$$\frac{d^2\theta}{dr^2} - \theta\frac{d\theta}{dr} - \frac{d\theta}{dr} = 0.$$

This equation also is satisfied by θ const., say c; then

$$\phi' = e^t = ae^{cr} = a\,(uv)^c.$$

But this is reducible by a translation of the surface to the general form $\phi = UV$, and hence belongs to the first class considered. We make an exception of this case and put $\dfrac{d\theta}{dr} = p$; the equation in p is

$$\frac{dp}{d\theta} - \theta - 1 = 0,$$

and consequently,

$$p = \frac{\theta^2 + 2\theta + \alpha}{2},$$

where α is a constant. From this we get

$$dr = \frac{2d\theta}{\theta^2 + 2\theta + \alpha}.$$

Three cases arise according as α is greater, equal to or less than unity.

1°. $\alpha > 1$. In this case

$$r + \beta = \frac{2}{\sqrt{\alpha - 1}}\tan^{-1}\frac{\theta + 1}{\sqrt{\alpha - 1}},$$

where β is a constant. Retracing the steps, it is readily found that

$$\phi = \frac{2\gamma}{\sqrt{\alpha - 1}}\tan\frac{\sqrt{\alpha - 1}\,(\log uv + \beta)}{2} + \delta.$$

If we replace γ by $\dfrac{\sqrt{\alpha - 1}}{2}\gamma$, $\dfrac{\sqrt{\alpha - 1}\,(\log u + \beta/2)}{2}$ by u, $\dfrac{\sqrt{\alpha - 1}\,(\log v + \beta/2)}{2}$ by v and δ by zero, this expression for ϕ becomes the same as (21′). Hence this case is the same as the general case where ϕ is a function of $U + V$.

2°. $\alpha = 1$. Now

$$r + \beta = -\frac{2}{\theta + 1},$$

and, consequently,

$$\phi = \frac{-\gamma}{\log uv + \beta} + \delta.$$

By a suitable choice of parameters this becomes

$$\phi = \frac{1}{u + v},$$

which has the sphere for the corresponding surface.

3°. $\alpha < 1$. Then

$$r + \beta = \frac{1}{\sqrt{1 - \alpha}} \log \frac{\theta + 1 - \sqrt{1 - \alpha}}{\theta + 1 + \sqrt{1 - \alpha}}.$$

From this it is found that ϕ takes the form

$$\phi = \frac{-\gamma}{\sqrt{1 - \alpha}\,[(uv)^{\sqrt{1 - \alpha}} e^{\beta\sqrt{1 - \alpha}} - 1]} + \delta.$$

By a convenient choice of parameters and a translation, this can be written

$$\phi = \frac{1}{uv - 1}.$$

Equation (13) becomes

$$2 \log uv + U - V = 0,$$

and, consequently,

$$U = -2(\log u + \delta), \qquad V = 2(\log v - \delta),$$

where δ is a constant. From (14) and (15) we get

$$z = -\frac{\log \frac{u}{v}(uv + 1) + 2\delta}{2(uv - 1)},$$

$$x - iy = -\tfrac{1}{4}\left[uv - \frac{1}{uv} + 2 \log uv + (uv + 1) \log^2 \frac{u}{v} - \frac{\left((uv + 1) \log \frac{u}{v} + 2\delta\right)^2}{uv - 1}\right].$$

In this case it is found that the equation of the lines of curvature is

$$\frac{du^2}{u^2} - \frac{dv^2}{v^2} = 0.$$

Hence if we put

$$\log u + \log v = u_1, \quad \log u - \log v = iv_1,$$

u_1 and v_1 are the real parameters of the lines of curvature. The cartesian coordinates have then the following expressions:

$$x + iy = \frac{1}{e^{u_1} - 1}, \quad x - iy = -\tfrac{1}{4}\Big[e^{u_1} - e^{-u_1} + 2u_1 - v_1^2(e^{u_1} + 1)$$
$$- \frac{[(e^{u_1} + 1)\, iv + 2\delta]^2}{e^{u_1} - 1}\Big], \quad z = -\frac{iv_1(e^{u_1} + 1) + 2\delta}{2(e^{u_1} - 1)}.$$

From this we see that the surface is imaginary. Since $\phi = \log UV$ belongs to the above class and leads to the minimal surfaces, we have the following theorem:

Minimal surfaces, the sphere and cylinders of revolution are the only real surfaces of constant mean curvature corresponding to the case where ϕ is a function of the product of any function of u by any function of v.

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